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CONFORMAL TRANSFORMATIONS FROM ONE MAP PROJECTION TO ANOTHER, USING DIVIDED DIFFERENCE INTERPOLATION

WITH A NOTE ON THE REMAINDER TERM

by F. YOUNG.

In his classic work (1) on the theory of conformal transformations, C.F. Gauss, showed that if in mapping one surface upon another, the ratio m of two infinitesimally small distances on the two surfaces can be expressed in the form

$$m^2 = \left(\frac{ds}{dS}\right)^2 = \frac{l[(du)^2 + (dv)^2]}{L[(dU)^2 + (dV)^2]} \qquad (1)$$

the necessary and sufficient condition that the transformation should be orthomorphic or conformal is that

$$u + iv = f(U + iV) \qquad (2)$$

In the study of map projections, it is usual to regard the mean sea level surface of the earth as a spheroid of revolution, in which case an element of length dS on this surface is given by

$$(dS)^2 = (N \cos\phi)^2 [(dq)^2 + (d\lambda)^2] \qquad (3)$$

where $N \cos\phi$ is the radius of a circle of parallel, q is the isometric latitude and λ is the longitude. If the corresponding element of length ds on the plane surface of the map is given by

$$(ds)^2 = (dx)^2 + (dy)^2 \qquad (4)$$

then the necessary and sufficient condition for a conformal transformation from the spheroidal surface to the plane is

$$x + iy = f(q + i\lambda) \qquad (5)$$

Sometimes, it is sufficiently accurate to regard the mean sea level surface of the earth as a sphere of radius R, and then the same condition for orthomorphism is obtained, provided that q is calculated as*

$\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) - \frac{4n}{1+n}(\sin\phi - \frac{1}{3}n \sin 3\phi + \frac{1}{5}n^2 \sin 5\phi + \ldots)$ in the first case and as $\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$ in the second case.

In each case, the rectangular coordinates of large numbers of points have been determined and used both in the cadastral survey and

(1) Gauss C.F. Collected Works 4 : 195 (Gottingen, 1873). The solution was obtained in 1822 and first published in ASTRONOMISCHE ABHANDLUNGE von H.C. Schumacher, Part 3 (Altona, 1825).

* ϕ is the geographical latitude and $n = \frac{a-b}{a+b}$ i.e. the ratio of the difference and sum of the semi-axes a and b of the meridian ellipse.

in the triangulation which forms the basis of maps of various kinds. The question that now arises is, having once chosen a particular functional relationship to give one kind of map projection and subsequently chosen another functional relationship to give a second projection, whether it is possible to transform rectangular coordinates directly from the one projection to the other without having to determine geographical coordinates on the spheroid or sphere. To take a specific example, suppose that the national survey of a country is computed first on the Transverse Mercator Projection and subsequently on the Lambert Conical Orthomorphic Projection, then if the rectangular coordinates of a certain number of points are known on both projections, can the coordinates of any new point on either projection be transformed directly to the corresponding coordinates on the other projection ?

If the functional relationships are of a fairly simple kind and are, for example, $x + iy = f(q + i\lambda)$ in the one case, and $X + iY = g(q + i\lambda)$ in the other case, then it may be possible to eliminate $(q + i\lambda)$ between these two equations and obtain a direct transformation of the form $X + iY = F(x + iy)$. But, in general, a transformation of this kind will be very difficult to obtain, especially if the earth is considered to be a spheroid of revolution rather than a sphere. It has, however, been done in at least one case in which a direct transformation between two adjacent strips of Gauss Conform Coordinates based on a spheroidal earth, has been effected(2) . Although the method does not depend directly on the elimination of $q + i\lambda$ between the two equations, this term is eliminated indirectly by writing down the general equations of transformation between the two systems and then finding the coefficients of the various terms in the one set of equations in terms of the coefficients of the corresponding terms in the other set. Sometimes, even if the equation of transformation is found, it may be so difficult to evaluate numerically that there may be little or no advantage over the usual calculation working through geographical coordinates.

In practice, cases have also arisen where little basic information is available about either or both of the projections. Perhaps the exact position of the origin, or the scale or the orientation is unknown or doubtful, or the latitude of a standard parallel or the longitude of a central meridian may be known to an insufficiently high degree of accuracy to make a direct transformation possible, even if it is theoretically possible to do so. In such cases, it is necessary to rely almost completely on the coordinates of stations common to the two systems. But then a further difficulty arises, in that observational errors complicate the problem, and the position of best fit obtained either by least squares or non-rigorous methods must be found. Generally, the solution of the problem in such cases is by no means easy, as the following extract from an article(3) by a well known geodesist in the United States Army Map Service will show :

(2) Lauf G.B. : A new method for the transformation of Gauss Conform Coordinates from one system to the next in South African Survey Journal, (1947) 7 : 86.

(3) O'Keefe J.A. : Approximate methods for datum adjustments in Transactions of the American Geophysical Union (1947). Vol 28, N° 4, page 519.

" In his Copenhagen prize paper, "Gauss set forth a more refined
" method for adjusting one survey to another by means of a confor-
" mal transformation of x - y coordinates. Gauss proposed that a
" complex polynomial be used; the coefficients would be determined
" from the differences in x and y of the two surveys on a few se-
" lected points. According to Thilo this method was tested by O.
" Schreiber using four points, in an attempt to reduce the triangu-
" lation of Mecklenburg to the Prussian system. However, the
" attempt failed, and the remaining cassures were considerably
" worse than with a simple blanket correction. An attempt along
" similar lines was made by the New York office of the United
" States Lake Survey. Instead of using only four points, the Lake
" Survey used all first-order junction points, and derived the best
" values of a complex polynomial of the second degree by least
" squares. The results, while better than Schreiber's, were not
" significantly better than those of Thilo, whose method amounted
" to the use of terms only up to the first degree. Extension to the
" third degree was carried out, but without significant improvement.
" The failure of the method, which was repeatedly attempted, could
" to some extent be traced. It is well known that if the line inte-
" gral of a complex function is taken around the boundary of some
" region, the integral around the boundary must vanish, or the
" function will cease to be analytic somewhere in the interior. In
" the case of the Lake Survey problems, it was found that the inte-
" gral, which could be roughly evaluated by the trapezoidal formula,
" did not vanish, and that the excess, no matter how distributed,
" would always lead to cassures of approximately the same amount
" as those obtained with the first degree polynomial alone. Of cour-
" se, by using a polynomial with as many terms as there are boun-
" dary points, agreement could be secured on these points; else-
" where, however, the polynomial would oscillate so badly that it
" would be worthless. The development of some mathematical
" technique sacrificing conformality to some extent for the sake of
" close agreement on the common points is urgently needed".

In spite of the apparently hopeless task involved, it has been found possible to effect a transformation of the kind mentioned in at least one case in South Africa. Using a complex polynomial of the fourth degree based on nine common stations, the coordinates of points on a local system (called "the Goldfields System") were transformed conformally to the Gauss Conform System (substantially the same as the Transverse Mercator System) using the method of least squares.[4] Over an area of about 250 squares miles* the average error on the nine common points was found to be 0.20** English foot, which is not significantly greater than the error in either of the original surveys, one of which was established as long ago as 1890. Recently, when another eight of the old Goldfields stations were re-determined on the Gauss Conform

(4) Lauf G.B. : The conformal transformation of Goldfields Coordinates into Gauss Conform Coordinates in South African Survey Journal. Vols 7 (1949) and 8 (1950).

* 1 square mile = 2.59 square kilometres (approx.).

** 1 English foot = 0.3048 metre (approx.).

System, it was found that the average difference between the values obtained from the field survey and those obtained by the conformal transformation method amounted to no more than 0.18 English foot, which confirms not only the accuracy of both the original surveys but also the method of transformation.

In this article a further method of transformation based on the application of the theory of divided difference interpolation to a table of complex numbers, will be described. To demonstrate the method, the geographical coordinates of a certain number of arbitrarily chosen points assumed to lie on a particular spheroid of reference are transformed conformally first on to one projection and then on to another. The two sets of coordinates are then used to form a table of complex numbers, and divided differences are calculated. By transforming the geographical coordinates of any new point on to the first system and then interpolating into the table of complex numbers, using the rectangular coordinates of this point as argument, the corresponding rectangular coordinates on the second system are obtained. The result is then compared with the coordinates obtained by direct transformation of the geographical coordinates to the second system. In each case, a complete check on the arithmetical work involved is obtained simply by repeating the interpolation process from the other end of the table. Once the table of complex numbers and divided differences have been drawn up, the coordinates of any number of points can be transformed directly from the one system to the other and fully checked in a few minutes. The transformation is rigorously conformal throughout.

The only difficulty in practice lies in evaluating the remainder term and this is particularly important because the accuracy of the method of interpolation depends upon it. In a Note appended to this article, a colleague has determined an upper-bound to the modulus of the remainder term in a transformation of a relatively simple kind based on a spherical earth and has shown that it is negligible in all practical cases. In other cases it may prove to be either too difficult or impossible to calculate the value of such a term. But if it can either be shown or justifiably assumed that in a particular transformation it is negligible, then the conversion can be effected with comparative ease, even in those cases where certain basic information about either or both projections is lacking. For instance, the method does not depend upon a knowledge of the latitudes of the standard parallels, the longitude of the central meridian, the scale factor nor upon the orientation of the two systems. It does not even require a knowledge of the unit of measurement used nor even of the type of projection employed. But it must be emphasised that in the strictest sense, if no information of any kind about the two projections is available, no unique solution of the problem is possible. Any arbitrarily chosen set of coordinates for the result (within limits) could be shown to be consistent with some projection, even if it were purely fanciful.

Suppose that the coordinates (x, y) of n points on one projection and (X, Y) of the same points on another projection are given, then the complex numbers representing these two sets of coordinates together with the corresponding divided differences can be set out in the form of a table as shown in Table I.

TABLE I

Table of Complex Numbers and Divided Differences

$z = x + iy$	$F(z) = Z = X + iY$	Divided Differences	
		1st order	2nd order
z_1	$F(z_1) = Z_1$	$[Z_1 Z_2]$	$[Z_1 Z_2 Z_3]$
z_2	$F(z_2) = Z_2$	$[Z_2 Z_3]$	....
z_3	$F(z_3) = Z_3$	....	
...	....	....	
z_{n-1}	$F(z_{n-1}) = Z_{n-1}$	$[Z_{n-1} Z_n]$	
z_n	$F(z_n) = Z_n$		

The divided differences of first order are defined as follows :

$$[Z_1 Z_2] = \frac{Z_1 - Z_2}{z_1 - z_2}$$

$$[Z_2 Z_3] = \frac{Z_2 - Z_3}{z_2 - z_3}$$

$$\text{........} \quad \text{......}$$

$$[Z_{n-1} Z_n] = \frac{Z_{n-1} - Z_n}{z_{n-1} - z_n}$$

In a similar way, second and higher order differences may be obtained :

$$[Z_1 Z_2 Z_3] = \frac{[Z_1 Z_2] - [Z_2 Z_3]}{z_1 - z_3}$$

$$[Z_1 Z_2 Z_3 \cdots Z_n] = \frac{[Z_1 Z_2 \cdots Z_{n-1}] - [Z_2 Z_3 \cdots Z_n]}{z_1 - z_n} \tag{6}$$

Assuming for the moment that the value of Z corresponding to z is known, and corresponding divided differences are formed,

$$[Z Z_1] = \frac{Z - Z_1}{z - z_1}$$

$$[Z Z_1 Z_2] = \frac{[Z Z_1] - [Z_1 Z_2]}{z - z_2}$$

$$[Z Z_1 Z_2 \cdots Z_n] = \frac{[Z Z_1 Z_2 \cdots Z_{n-1}] - [Z_1 Z_2 \cdots Z_n]}{z - z_n}$$

then by repeated substitution, the following equations are obtained :

$$Z = Z_1 + (z - z_1)[Z Z_1]$$

$$Z = Z_1 + (z - z_1)[Z_1 Z_2] + (z - z_1)(z - z_2)[Z Z_1 Z_2]$$

$$Z = Z_1 + (z - z_1)[Z_1 Z_2] + (z - z_1)(z - z_2)[Z_1 Z_2 Z_3] + \cdots + (z - z_1)(z - z_2)\cdots(z - z_{n-1})[Z_1 Z_2 \cdots Z_n] + R_n(Z)$$

where

$$R_n(Z) = (z - z_1)(z - z_2)\cdots(z - z_n)[Z Z_1 Z_2 \cdots Z_n] \tag{7}$$

This is Newton's divided difference interpolation formula applied to complex numbers.

Four numerical examples have been chosen to illustrate the method.

Example I. In this example, the earth is assumed to be a sphere of radius R = 6 371 227.711 mètres, this value being the radius of a sphere, whose surface area is equal to that of the surface of the International spheroid. Points on this surface are transformed to the plane using first Mercator's projection and secondly the Stereographic projection (equatorial case). In Mercator's projection, the law of transformation is $x + iy = R(q + i\lambda)$ where $q = \log \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$

so that $x = R \log \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$ and $y = R\lambda$ (8)

For the Stereographic projection, the basic equation is $X + iY = 2R \tanh \frac{1}{2}(q + i\lambda)$ *. To express X and Y in terms of geographical coordinates, we note from (8) that $e^q = \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$ and so $e^{-q} = \cot\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$

Hence $\cosh q = \frac{1}{2}(e^q + e^{-q}) = \sec \phi$ and similarly

$$\sinh q = \frac{1}{2}(e^q - e^{-q}) = \tan \phi$$

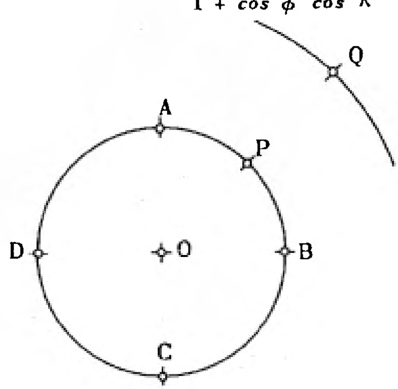
Then
$$(X + iY) = 2R \frac{\sinh \frac{1}{2}(q + i\lambda) \ \sinh \frac{1}{2}(q - i\lambda)}{\cosh \frac{1}{2}(q + i\lambda) \ \cosh \frac{1}{2}(q - i\lambda)}$$

$$= 2R \frac{\sinh q \ \sinh i\lambda}{\cosh q \ \cosh i\lambda}$$

$$= 2R \frac{\tan \phi + i \sin \lambda}{\sec \phi + \cos \lambda}$$

$$= 2R \frac{\sin \phi + i \cos \phi \sin \lambda}{1 + \cos \phi \cos \lambda}$$

and hence
$$X = \frac{2R \sin \phi}{1 + \cos \phi \cos \lambda} \quad \text{and} \quad Y = \frac{2R \cos \phi \sin \lambda}{1 + \cos \phi \cos \lambda} \qquad (9)$$



Four points A, B, C and D are now chosen, such that when transformed to Mercator's projection, they lie at approximately equal distances along the circumference of a circle of about 50 miles (**) radius. In addition, three other points O, P and Q are chosen, such that O lies near the centre of the circle and P and Q approximately along the bisector of the angle AOB, P at a distance of about 50 miles and Q about 100 miles from O.

* Dans la notation de l'auteur les expressions *sinh* x, *cosh* x, *tanh* x représentent les fonctions hyperboliques *ch* x, *sh* x, *th* x.
** 1 mile = 1.609 kilomètres (approx.).

The adopted values of the geographical coordinates of these points are given in Table II.

TABLE II

Point	Latitude				Longitude			
A	30°.	37'.	40"	South	25°.	0'.	0"	East
B	30	0	0		25	43	30	
C	29	22	20		25	0	0	
D	30	0	0		24	16	30	
O	30	0	0		25	0	0	
P	30	27	20		25	31	30	
Q	30	53	40		26	1	30	

These coordinates are then transformed to the two different projections using equations (8) and (9). From the two sets of coordinates of A, B, C and D, the table of complex numbers and divided differences is set up. In turn, the coordinates of O, P and Q are transformed from Mercator's projection to the Stereographic projection, using Newton's divided difference interpolation formula up to and including third order differences, that is $R_4(Z)$ is neglected. To check the calculation, the interpolation is also carried out from the bottom of the table upwards using the formula

$$Z = Z_4 + (z-z_4)[Z_3 Z_4] + (z-z_4)(z-z_3)[Z_2 Z_3 Z_4] + (z-z_4)(z-z_3)(z-z_2)[Z_1 Z_2 Z_3 Z_4] + R_4(Z) \quad (10)$$

where $R_4(Z)$ is one again neglected. The two sets of transformed coordinates should in each case be identical apart from rounding off errors in the calculation. In the example, the values all agree to within $\pm$ 0.001 metre. The accuracy of the method may then be judged by a comparison of the results obtained by divided difference interpolation and those obtained by using the basic formulae of the projection, the differences of the two sets of results being the value of $R_4(Z)$ at the three points. In this example, the actual values of $|R_4(Z)|$ are 4 mms at O, 12 mms at P and 79 mms at Q. In the Note appended to this article it is shown that the value of $|R_4(Z)|$ cannot exceed 25 mms at N, 52 mms at P and 437 mms at Q. It must, however, be emphasised that the latter values represent the upper bounds of $|R_4(Z)|$, which does not mean that $|R_4(Z)|$ will necessarily reach values of this magnitude at any point within the area considered. Finally, it should be remembered that in practice, points will generally be chosen within the circle passing through or near the pivotal points A, B, C and D and so it would be unlikely that the transformation of the coordinates of a point, such as Q, 50 miles beyond the oircumference of this circle, should be necessary.

Example II. In this example the same two projections based on a spherical earth of the same radius are used. But instead of four points approximately equally spaced around the circumference of a circle of 50 miles radius, five points are chosen and the radius of the circle is increased to about 150 miles. This means that the points are far enough apart to control an area as big as Basutoland and the Orange Free State combined. For purposes of reference, the geographical coordinates of the five pivotal points A, B, C, D and E and the two check points O (near the centre of the circle) and P (near the circumference of the circle between D and E) are given in Table III.

TABLE III

Point	Latitude			Longitude		
A	30°.	2'.	0" South	27°.	20'.	30" East
B	29	9	20	28	43	10
C	27	44	0	28	11	30
D	27	44	0	26	29	30
E	29	9	20	25	57	50
O	28	45	40	27	20	30
P	28	22	0	25	57	50

As a result of the transformation, the two sets of coordinates of O and P obtained by interpolation from the top and bottom of the table agree to within ± 1 millimetre and these values in turn agree with the values calculated from the basic formulae of the projection to ± 0 mms for point O and to - 1 mm for point P.

Example III. In this case, four trigonometrical stations about 7 miles apart were chosen. The first projection is the Gauss Conform Projection with central meridian in longitude 29°; the system is 2° wide and the coordinates are in Cape Roods (*). The second projection is the Universal Transverse Mercator System, Zone 35, with 6° belts and the coordinates are in metres. Both projections are based on the Clarke 1880 spheroid of reference. As a result of the transformation the two sets of coordinates agree to within ± 1 mm.

Example IV. For this example, four pivotal points A, B, C, and D were chosen, roughly at the corners of a rectangle, and sufficiently far apart to command an area as large as the whole of Switzerland. In the first case, the geographical coordinates of the four pivotal points were transformed to the Lambert Conical Orthomorphic Projection with central meridian in longitude 8° 15' E and with a standard parallel in latitude 45° 54' N and scale factor along this parallel of 0.998 992 911. In the second case, the geographical coordinates were transformed to the Transverse Mercator Projection, with central meridian in longitude 7° 15' East and a scale factor of unity along the meridian. Both projections are on the Bessel Spheroid of Reference and the unit of measurement is the metre. The geographical coordinates of the four pivotal points A, B, C and D and the four check points, O, P, Q and R are given in Table IV.

* 1 Cape Rood = 3.778 297 metres (approx.).

TABLE IV

Point	Latitude			Longitude		
A	47°.	30'.	0" North	7°.	0'.	0" East
B	47	30	0	9	30	0
C	46	0	0	9	30	0
D	46	0	0	7	0	0
0	46	45	0	8	15	0
P	47	30	0	7	15	0
Q	46	10	0	7	20	0
R	46	10	0	9	0	0

After the transformation using the interpolation method, the errors in the results were found to be :

Point	Δ X	Δ Y
0	+ 3 mms	- 39 mms
P	+ 14 "	- 15 "
Q	- 3 "	- 28 "
R	+ 12 "	- 32 "

As the projection tables for the Lambert Conical Orthomorphic Projection were originally designed to give an accuracy of ± 0.01 metre, the result obtained by the transformation method appears to be accurate to about ± 4 units.

Transformation from Mercator's Projection to the Stereographic Projection using a spherical earth (Four pivotal points).

Stn	Mercator's Projection x (metres)	y	Stereographic Projection (equatorial case) X (metres)	Y	First Order Divided Differences $[Z_r\ Z_{r+1}]$		Second Order Divided Differences $[Z_r\ Z_{r+1}\ Z_{r+2}] \times 10^8$		Third Order Divided Differences $[Z_r\ Z_{r+1}\ Z_{r+2}\ Z_{r+3}] \times 10^{15}$	
A	3 580 619.757	2 779 972.524	3 647 312.248	2 603 518.565	0.96275 24897	- 0.11780 10355	- 2.31046 21	- 1.31155 56	- 1.68136	+ 0.89175
B	3 499 754.529	2 860 591.727	3 578 956.074	2 690 660.911	0.96647 74284	- 0.11568 65389	- 2.28967 65	- 1.30521 18		
C	3 419 399.251	2 779 972.524	3 491 967.955	2 622 040.295	0.96437 29257	- 0.11199 47010				
D	3 499 754.529	2 699 353.321	3 560 431.486	2 535 293.953						

	Coordinates of stations to be transformed		Results obtained from the transformation		Correct results		Errors	
O	3 499 754.529	2 779 972.524	3 569 544.085	2 612 893.066	3 569 544.081	2 612 893.065	- 0.004	- 0.001
P	3 558 383.806	2 838 351.947	3 632 833.088	2 662 251.361	3 632 833.076	2 662 251.358	- 0.012	- 0.003
Q	3 615 127.888	2 893 951.397	3 694 145.466	2 708 847.978	3 694 145.390	2 708 847.957	- 0.076	- 0.021

Calculation of Coordinates on Stereographic Projection

	X (metres)	Y
Station O Z_1 =	3 647 312.248	2 603 518.565
$(z-z_1)\ [Z_1\ Z_2]$ =	- 77 853.200	+ 9 526.008
$(z-z_1)(z-z_2)\ [Z_1\ Z_2\ Z_3]$ =	+ 85.504	- 150.626
$(z-z_1)(z-z_2)(z-z_3)\ [Z_1\ Z_2\ Z_3\ Z_4]$ =	- 0.467	- 0.881
O =	3 569 544.085	2 612 893.066
Station P Z_1 =	3 647 312.248	2 603 518.565
$(z-z_1)\ [Z_1\ Z_2]$ =	- 14 530.561	+ 58 824.353
$(z-z_1)(z-z_2)\ [Z_1\ Z_2\ Z_3]$ =	+ 51.500	- 90.437
$(z-z_1)(z-z_2)(z-z_3)\ [Z_1\ Z_2\ Z_3\ Z_4]$ =	- 0.099	- 1.119
P =	3 632 833.088	2 662 251.362
Station Q Z_1 =	3 647 312.248	2 603 518.565
$(z-z_1)\ [Z_1\ Z_2]$ =	+ 46 649.618	+ 105 668.350
$(z-z_1)(z-z_2)\ [Z_1\ Z_2\ Z_3]$ =	+ 183.433	- 322.774
$(z-z_1)(z-z_2)(z-z_3)\ [Z_1\ Z_2\ Z_3\ Z_4]$ =	+ 0.167	- 6.163
Q =	3 694 145.466	2 708 847.978

Checks

	X (metres)	Y
Station O Z_4 =	3 560 431.486	2 535 293.953
$(z-z_4)\ [Z_3\ Z_4]$ =	+ 9 028.924	+ 77 746.977
$(z-z_3)(z-z_4)\ [Z_2\ Z_3\ Z_4]$ =	+ 84.554	- 148.329
$(z-z_2)(z-z_3)(z-z_4)\ [Z_1\ Z_2\ Z_3\ Z_4]$ =	- 0.878	+ 0.466
O =	3 569 544.086	2 612 893.067
Station P Z_4 =	3 560 431.486	2 535 293.953
$(z-z_4)\ [Z_3\ Z_4]$ =	+ 72 107.597	+ 127 480.343
$(z-z_3)(z-z_4)\ [Z_2\ Z_3\ Z_4]$ =	+ 296.047	- 521.147
$(z-z_2)(z-z_3)(z-z_4)\ [Z_1\ Z_2\ Z_3\ Z_4]$ =	- 2.042	- 1.788
P =	3 632 833.088	2 662 251.361
Station Q Z_4 =	3 560 431.486	2 535 293.953
$(z-z_4)\ [Z_3\ Z_4]$ =	+ 133 056.897	+ 174 743.911
$(z-z_3)(z-z_4)\ [Z_2\ Z_3\ Z_4]$ =	+ 659.572	- 1 178.441
$(z-z_2)(z-z_3)(z-z_4)\ [Z_1\ Z_2\ Z_3\ Z_4]$ =	- 2.488	- 11.445
Q =	3 694 145.467	2 708 847.978

Transformation from Mercator's Projection to the Stereographic Projection, using a spherical earth (Five pivotal points).

Stn	Mercator's Projection x (metres)	y	Stereographic Projection (equatorial case) X (metres)	Y	First Order Divided Differences $[Z_r\ Z_{r+1}]$		Second Order Divided Differences $[Z_r\ Z_{r+1}\ Z_{r+2}] \times 10^8$		Third Order Divided Differences $[Z_r\ Z_{r+1}\ Z_{r+2}\ Z_{r+3}] \times 10^{15}$		Fourth Order Divided Differences $[Z_r\ Z_{r+1}\ Z_{r+2}\ Z_{r+3}\ Z_{r+4}] \times 10^{22}$	
A	3 504 035.296	3 040 363.285	3 605 191.375	2 864 164.673	0.97538 94544	-0.12918 08734	-2.333 0326	-1.513 5493	-1.794 302	+0.986 136	+1.01610	+0.40717
B	3 391 781.487	3 193 570.660	3 515 491.656	3 028 102.576	0.98363 53911	-0.12696 45960	-2.271 2959	-1.525 3971	-1.799 470	+0.965.998		
C	3 211 911.472	3 134 882.351	3 331 113.806	2 993 211.802	0.98394 19594	-0.11859 42633	-2.241 6963	-1.470 2587				
D	3 211 911.472	2 945 844.219	3 308 694.968	2 807 209.252	0.97626 76003	-0.11568 55433						
E	3 391 781.487	2 887 155.910	3 477 506.847	2 729 105.397								

Stn	Coordinates of stations to be transformed		Results obtained from the transformation		Correct results		Errors	
O	3 341 651.967	3 040 363.285	3 446 982.477	2 884 436.891	3 446 982.477	2 884 436.891	± 0.000	± 0.000
P	3 291 711.512	2 887 155.910	3 380 073.386	2 740 664.165	3 380 073.385	2 740 664.165	- 0.001	± 0.000

Calculation of Coordinates on Stereographic Projection

		X (metres)	Y
Station O	Z_1 =	3 605 191.375	2 864 164.673
	$(z-z_1)\ [Z_1\ Z_2]$ =	- 158 386.987	+ 20 976.820
	$(z-z_1)\ (z-z_2)\ [Z_1\ Z_2\ Z_3]$ =	+ 186.632	- 703.625
	$(z-z_1)\ (z-z_2)\ (z-z_3)\ [Z_1\ Z_2\ Z_3\ Z_4]$ =	- 8.538	- 1.051
	$(z-z_1)\ (z-z_2)\ (z-z_3)\ (z-z_4)\ [Z_1\ Z_2\ Z_3\ Z_4\ Z_5]$ =	- 0.005	+ 0.074
	O =	3 446 982.477	2 884 436.891
Station P	Z_1 =	3 605 191.375	2 864 164.673
	$(z-z_1)\ [Z_1\ Z_2]$ =	- 226 889.842	- 122 008.686
	$(z-z_1)\ (z-z_2)\ [Z_1\ Z_2\ Z_3]$ =	+ 1 816.289	- 1 486.591
	$(z-z_1)\ (z-z_2)\ (z-z_3)\ [Z_1\ Z_2\ Z_3\ Z_4]$ =	- 44.658	- 5.317
	$(z-z_1)\ (z-z_2)\ (z-z_3)\ (z-z_4)\ [Z_1\ Z_2\ Z_3\ Z_4\ Z_5]$ =	+ 0.222	+ 0.086
	P =	3 380 073.386	2 740 664.165

Checks

		X (metres)	Y
Station O	Z_5 =	3 477 506.847	2 729 105.397
	$(z-z_5)\ [Z_4\ Z_5]$ =	- 31 215.948	+ 155 370.657
	$(z-z_4)\ (z-z_5)\ [Z_3\ Z_4\ Z_5]$ =	+ 692.999	- 30.839
	$(z-z_3)\ (z-z_4)\ (z-z_5)\ [Z_2\ Z_3\ Z_4\ Z_5]$ =	- 1.489	- 8.351
	$(z-z_2)\ (z-z_3)\ (z-z_4)\ (z-z_5)\ [Z_1\ Z_2\ Z_3\ Z_4\ Z_5]$ =	+ 0.068	+ 0.027
	O =	3 446 982.477	2 884 436.891
Station P	Z_5 =	3 477 506.847	2 729 105.397
	$(z-z_5)\ [Z_4\ Z_5]$ =	- 97 695.074	+ 11 576.649
	$(z-z_4)\ (z-z_5)\ [Z_3\ Z_4\ Z_5]$ =	+ 265.360	- 14.245
	$(z-z_3)\ (z-z_4)\ (z-z_5)\ [Z_2\ Z_3\ Z_4\ Z_5]$ =	- 3.835	- 3.613
	$(z-z_2)\ (z-z_3)\ (z-z_4)\ (z-z_5)\ [Z_1\ Z_2\ Z_3\ Z_4\ Z_5]$ =	+ 0.088	- 0.023
	P =	3 380 073.386	2 740 664.165

Transformation from the Gauss Conform System to the Universal Transverse Mercator System, using Clarke's 1880 Spheroid of Reference

Stn	Gauss Conform System L_o 29° ; 2° belt (Cape Roods) y	x	Universal Transverse Mercator System Zone 35° ; 6° belt (Metres) E	N
Constant	+ 20 000.000	+ 750 000.000	+ 600 000.000	+ 7 100 000.000
Townlands	+ 4 383.284	+ 5 080.095	+ 8 443.84	+ 48 122.86
Zwartkop	+ 3 088.760	+ 6 313.278	+ 13 261.86	+ 43 391.43
Constantia	+ 4 264.902	+ 7 873.522	+ 8 730.44	+ 37 566.75
Schurveberg	+ 6 369.462	+ 5 906.865	+ 895.80	+ 45 114.60

First Order Divided Differences $[Z_r Z_{r+1}]$		Second Order Divided Differences $[Z_r Z_{r+1} Z_{r+2}] \times 10^6$		Third Order Divided Differences $[Z_r Z_{r+1} Z_{r+2} Z_{r+3}] \times 10^{12}$	
- 3.776 5175	+ 0.057 3901	+ 0.03500	- 0.00030	+ 0.184	+ 0.079
- 3.776 5208	+ 0.057 4879	+ 0.03530	+ 0.00001		
- 3.776 4050	+ 0.057 4736				

	Coordinates of stations to be transformed		Result obtained from the transformation	
Mooiplaats Rd.	+ 4 719.441	+ 6 286.865	+ 7 105.103	+ 43 584.833

Result by working through geographicals		Error		Error in tables		Residual error	
7 105.103	43584.904	0.000	+ 0.071	0.000	+ 0.072	0.000	- 0.001

Calculation of Coordinates on Universal Transverse Mercator System

	E (metres)	N
Z_1 =	+ 8 443.84	+ 48 122.86
$(z-z_1)\,[Z_1 Z_2]$ =	- 1 338.759	- 4 538.096
$(z-z_1)(z-z_2)\,[Z_1 Z_2 Z_3]$ =	+ 0.021	+ 0.068
$(z-z_1)(z-z_2)(z-z_3)\,[Z_1 Z_2 Z_3 Z_4]$ =	+ 0.001	± 0.000
Mooiplaats Rd.	+ 7 105.103	+ 43 584.832

Check

	E (metres)	N
Z_4 =	+ 895.80	+ 45 114.60
$(z-z_4)\,[Z_3 Z_4]$ =	+ 6 209.308	- 1 529.867
$(z-z_3)(z-z_4)\,[Z_2 Z_3 Z_4]$ =	- 0.005	+ 0.099
$(z-z_2)(z-z_3)(z-z_4)\,[Z_1 Z_2 Z_3 Z_4]$ =	± 0.000	+ 0.001
Mooiplaats Rd.	+ 7 105.103	+ 43 584.833

Transformation from the Lambert Conical Orthomorphic Projection to the Transverse Mercator Projection, using Bessel's Spheroid of Reference

Stn	Lambert Conical Orthomorphic Projection x (metres)	y	Transverse Mercator Projection X (metres)	Y	First Order Divided Differences [Z_r Z_{r+1}]		Second Order Divided Differences [Z_r Z_{r+1} Z_{r+2}] × 10^9		Third Order Divided Differences [Z_r Z_{r+1} Z_{r+2} Z_{r+3}] × 10^{15}	
A	705 893.728	779 425.377	481 166.401	5 262 329.044	1.00063 62275	0.01287 92744	1.319 602	1.562 940	4.171 037	0.108 688
B	894 106.272	779 425.377	669 498.691	5 264 753.085	1.00114 83683	0.01295 77185	1.326 820	0.867 907		
C	896 716.076	612 860.253	674 269.796	5 098 030.500	1.00103 97444	0.01257 11004				
D	703 283.924	612 860.253	480 636.524	5 095 598.845						

	Coordinates of stations to be transformed		Results obtained from the transformation		Correct results		Errors	
O	800 000.000	695 381.791	576 400.524	5 179 413.815	576 400.527	5 179 413.776	+ 0.003	- 0.039
P	724 713.873	779 159.988	499 999.986	5 262 298.765	500 000.000	5 262 298.750	+ 0.014	- 0.015
Q	729 286.138	631 013.708	506 435.088	5 114 095.287	506 435.085	5 114 095.259	- 0.003	- 0.028
R	857 857.217	630 879.419	635 135.958	5 115 580.840	635 135.970	5 115 580.808	+ 0.012	- 0.032

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NOTE ON THE REMAINDER TERM

1. Suppose we map points (q, λ) on the earth's surface on to two planes (which we may call a z-plane, where $z = x + iy$, and a Z-plane, where $Z = X + iY$) using two different transformations

$$z = \alpha(q + i\lambda)$$
$$Z = \beta(q + i\lambda)$$

If it is possible to eliminate q and λ between these two equations, then we can find a relation $Z = f(z)$ which can be used to map points from the z-plane on to the Z-plane. In general, however, it is very difficult to find the explicit function $f(z)$, or it may be inconvenient to calculate $f(z)$ for a large number of values of z , so that surveyors in order to find the Z corresponding to a given z would first have to find the point (q, λ) (using the first of the above transformations) and then calculate Z (using the second). This often involves a great deal of heavy and complicated arithmetic.

Professor Lauf has discovered that excellent results involving a great saving of work can be obtained by using the Newton Divided Difference Formula.

2. Newton's Divided Difference Formula (involving complex values of the variable)

Let $f(t)$ be a regular function of t on and within a simple closed contour C, enclosing the points $z_1, z_2, \cdots z_n, z$.

Then
$$f(z) = f(z_1) + (z-z_1)[Z_1 Z_2] + (z-z_1)(z-z_2)[Z_1 Z_2 Z_3] + \cdots\cdots + (z-z_1)\cdots(z-z_{n-1})[Z_1 Z_2 \cdots Z_n] + R_n(Z),$$

where
$$[Z_1 Z_2 \cdots Z_r] = \frac{1}{2i\pi}\int_c \frac{f(t)\,dt}{(t-z_1)(t-z_2)\cdots(t-z_r)}$$
$$= \sum_{s=1}^{r} \frac{f(z_s)}{(z_s-z_1)\cdots(z_s-z_{s-1})(z_s-z_{s+1})\cdots(z_s-z_r)}$$

and
$$R_n(Z) = \frac{N}{2i\pi}\int_c \frac{f(t)\,dt}{(t-z_1)\cdots(t-z_n)(t-z)},$$

N being the product $(z-z_1)\cdots(z-z_n)$

Proof

Since (i) $\frac{1}{t-z} = \frac{1}{t-z_1} + \frac{z-z_1}{t-z_1} \cdot \frac{1}{t-z}$

and (ii) $\frac{z-z_{r+1}}{t-z_{r+1}} \cdot \frac{1}{t-z} - \frac{1}{t-z} + \frac{1}{t-z_{r+1}} = 0$

it follows (by induction) that

$$\frac{1}{t-z} = \frac{1}{t-z_1} + \frac{z-z_1}{t-z_1} \cdot \frac{1}{t-z_2} + \frac{(z-z_1)(z-z_2)}{(t-z_1)(t-z_2)} \cdot \frac{1}{(t-z_3)} + \cdots\cdots$$
$$+ \frac{(z-z_1)(z-z_2)\cdots(z-z_{n-1})}{(t-z_1)(t-z_2)\cdots(t-z_{n-1})} \frac{1}{(t-z_n)} + \frac{(z-z_1)(z-z_2)\cdots(z-z_n)}{(t-z_1)(t-z_2)\cdots(t-z_n)} \frac{1}{(t-z)}.$$

We next multiply throughout by $\frac{1}{2i\pi} f(t)$ and integrate both sides of the identity about the contour C . The formula follows by Cauchy's Integral Theorem (1) .

3. If now $|R_n(Z)|$ is very small for any particular Z and a particular value of n , we can neglect this Remainder Term and assume that

$$f(z) = f(z_1) + \sum_{m=2}^{n} (z-z_1)\cdots(z-z_{m-1}) [Z_1 Z_2 \cdots Z_m] \tag{A}$$

Professor Lauf has found that starting with four fixed points z_1, z_2, z_3 and z_4 evenly distributed on a circle of radius 50 miles, then $|R_4(Z)|$ is generally negligible provided z is a point within 100 miles from the centre.

The calculation involves the determination of the divided differences $[Z_1 Z_2]$, $[Z_1 Z_2 Z_3]$ and $[Z_1 Z_2 Z_3 Z_4]$ (which are independent of Z). And the terms of the above expansion (A) can then easily be computed on a small calculating machine.

4. It is, of course, not easy to justify the method in all cases. We shall do so in the case where the points z are mapped by Mercator's Projection $z = R(q+i\lambda)$ and the points Z by the Stereographic Projection (equatorial case),

$$Z = 2R \; tanb \; \frac{1}{2}(q + i\lambda)$$

It then follows that

$$Z = 2R \; tanb \; \frac{z}{2R}$$

that is,

$$(X + iY) = 2R \frac{sinb \frac{x}{R} + i \; sin \frac{y}{R}}{cosb \frac{x}{R} + cos \frac{y}{R}} \tag{B}$$

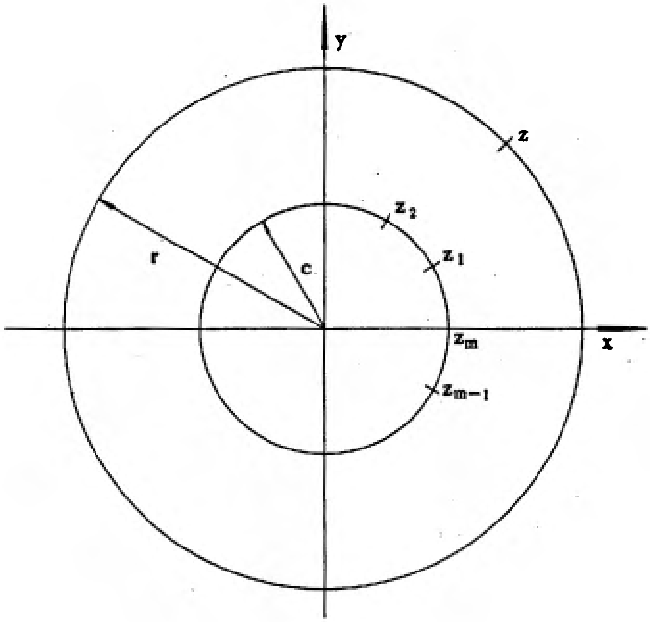
In this case Z can be computed directly, when z is know, although the hyperbolic and circular functions would have to be found to a large number of decimal places in view of the large value of the factor 2R .

(1) Whittaker and Watson : Modern Analysis .

Our object is not, however, to point out the advantages of using the Difference Formula (A) in this particular instance but rather to demonstrate that the formula is as reliable as (B).

This gives rise to the hope that (A) produces reliable results in other cases where (B) is either not known or difficult to handle.

5. We need two preliminary lemmas



(i) Let m points $z_1, z_2 \dots z_m$ be uniformly distributed about a circle of centre O and radius c and let z be a point on a concentric circle of radius r.

Then $|(z-z_1)(z-z_2)\cdots(z-z_m)|$ does not exceed $r^m + c^m$

We can take the point z_r to be $c\,e^{i\frac{2\pi}{m}r}$

$$\left|(z-ce^{i\frac{2\pi}{m}})\cdots(z-ce^{i2\pi})\right| = |z^m - c^m|$$

$$\leqslant |z|^m + c^m$$

$$= r^m + c^m$$

(ii) If $\frac{\pi}{2} < a < \pi$ then the maximum value of

$\left|\tanh\left(\frac{a}{2}e^{i\theta}\right)\right|$ is $\tan\frac{a}{2}$

Let $g = | \tanh (\frac{a}{2} e^{i\theta}) |^2$

$= \tanh (\frac{a}{2} e^{i\theta}) \tanh (\frac{a}{2} e^{-i\theta})$

then $\frac{dg}{d\theta} = \frac{1}{\alpha} [- \frac{ia}{2} e^{-i\theta} \sinh (a e^{i\theta}) + \frac{ia}{2} e^{i\theta} \sinh (a e^{-i\theta})]$

where $\alpha = 2 \cosh^2 (\frac{1}{2} a e^{i\theta}) \cosh^2 (\frac{1}{2} a e^{-i\theta})$

$\frac{dg}{d\theta} = 0$, when $\theta = \frac{\pi}{2}$ and $g = \tan^2 \frac{a}{2}$

$$\frac{d^2 g}{d\theta^2} = \frac{a (a \cos a - \sin a)}{2 \cos^4 \frac{a}{2}} < 0 \text{ for } \frac{\pi}{2} < a < \pi$$

Hence a maximum at $\theta = \frac{\pi}{2}$

When g is a maximum, so also is $\sqrt{g}$

6. Suppose now that S is a region covered by a circle of radius 100 miles. We can choose suitable x and y axes through the centre of the circle and select four points K, L, M and N on the axes each at a distance of, say, 50 miles from the centre.

The transformation $Z = 2R \tanh \frac{z}{2R}$ maps the region S (in the z-plane) on to another region T (in the Z-plane).

By starting with the four points as base and using the formula (A) with $n = 4$ we find that the error in computing $f(z)$ does not exceed

$$|R_4(Z)| = |(z-z_1)(z-z_2)(z-z_3)(z-z_4)| \left| \frac{1}{2\pi i} \int_C \frac{2R \tanh \frac{t}{2R} \, dt}{(t-z_1)(t-z_2)(t-z_3)(t-z_4)(t-z)} \right|$$

where : C is the circle $|t| = aR$, $\frac{\pi}{2} < a < \pi$ and z_1 , z_2 , z_3 and z_4 are the points , K, L, M and N.

Applying Cauch'ys Inequality Theorem[2] $| \int_C F(z) \, dz | < ML$ where $| F(z) | < M$ on the path C and L is the length of the path, and taking $a = 2{\cdot}48$ and $R = 3959$ miles the error involved does not exceed 1.44 feet (= 437 mms), when z lies within the circle $| z | \leqslant 100$ miles.

N.B. $|(z-z_1)(z-z_2)(z-z_3)(z-z_4)| \leqslant |z|^4 + c^4$

$< (100^4 + 50^4) \text{ miles}^4$

$< \frac{17}{16} 10^8 \text{ miles}^4$

(2) Whittaker and Watson : Modern Analysis .

Let $g = | \tanh (\frac{a}{2} e^{i\theta}) |^2$

$$= \tanh (\frac{a}{2} e^{i\theta}) \tanh (\frac{a}{2} e^{-i\theta})$$

then $\frac{dg}{d\theta} = \frac{1}{\alpha} [- \frac{ia}{2} e^{-i\theta} \sinh (a e^{i\theta}) + \frac{ia}{2} e^{i\theta} \sinh (a e^{-i\theta})]$

where $\alpha = 2 \cosh^2 (\frac{1}{2} a e^{i\theta}) \cosh^2 (\frac{1}{2} a e^{-i\theta})$

$\frac{dg}{d\theta} = o$, when $\theta = \frac{\pi}{2}$ and $g = \tan^2 \frac{a}{2}$

$$\frac{d^2 g}{d\theta^2} = \frac{a (a \cos a - \sin a)}{2 \cos^4 \frac{a}{2}} < o \quad \text{for } \frac{\pi}{2} < a < \pi$$

Hence a maximum at $\theta = \frac{\pi}{2}$

When g is a maximum, so also is $\sqrt{g}$

6. Suppose now that S is a region covered by a circle of radius 100 miles. We can choose suitable x and y axes through the centre of the circle and select four points K, L, M and N on the axes each at a distance of, say, 50 miles from the centre.

The transformation $Z = 2R \tanh \frac{z}{2R}$ maps the region S (in the z-plane) on to another region T (in the Z-plane).

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$$|R_4 (Z)| = |(z-z_1)(z-z_2)(z-z_3)(z-z_4)| \, \left| \frac{1}{2\pi i} \int_C \frac{2R \tanh \frac{t}{2R} \, dt}{(t-z_1)(t-z_2)(t-z_3)(t-z_4)(t-z)} \right|$$

where : C is the circle $|t| = aR$, $\frac{\pi}{2} < a < \pi$ and z_1 , z_2 , z_3 and z_4 are the points , K, L, M and N.

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$$< (100^4 + 50^4) \text{ miles}^4$$

$$< \frac{17}{16} 10^8 \text{ miles}^4$$

(2) Whittaker and Watson : Modern Analysis .