9. TRANSFORMATION OF COORDINATES FROM ONE MAP PROJECTION TO ANOTHER

It frequently happens that the coordinates of a station are known on one map projection and are required on another projection. Alternatively, they may be known on one *zone* of a projection and are required on the adjacent zone of the same projection. When full details are known of both projections, and projection tables or computer programmes are available, it is usually a relatively simple matter to transform the rectangular coordinates on the one projection, first to geodetic coordinates, latitude and longitude, and then to transform these coordinates to rectangular coordinates on the other projection. Indeed, this is the method currently used for transforming rectangular coordinates from one zone to the adjacent zone on the Australian Map Grid.

However, cases do arise where relatively little information is known about either or both projections. For instance, the exact unit of measure used in one of the projections may not be known, or the orientation of the one projection relative to the other projection may be doubtful or simply not known. Then again, ambiguities may exist as to the value of the standard parallel or central meridian used in the two projections. Such cases arise most frequently in the transformation from a map projection, introduced at a much earlier date, to a projection introduced quite recently and where the earlier records are no longer available. If both projections are known to be orthomorphic transformations from the same spheroid and the rectangular coordinates of several common stations on both projections are known, then the transformation from the one projection to the other can be effected quite easily regardless of the fact that not even the name of either projection need be known.

9.1 Transformation of Gauss Conform Coordinates from One Zone to the Adjacent Zone

In the literature on the subject, several methods have been developed for transforming Gauss Conform Coordinates from one zone to the adjacent zone.

In 1938, W. Hristow published an article in the *Zeitschrift für Vermessungswesen* (pages 534 to 540) entitled 'Über die Transformation zwischen zwei Gauss-Krügerschen Streifen'. A second article by the same author appeared in the same periodical, 1941 (pages 283 to 287). This was entitled 'Allgemeine Formeln zur Transformation zwischen zwei Gauss-Krügerschen Streifen'. Another method due to A. Hirvonen was published in the same periodical, 1938 (pages 321 to 326) and entitled 'Transformation der Gauss-Krügerschen Koordinaten von einem Streifen zu dem benachbarten'.

In 1959, the authors Jordan and Eggert published in their Handbuch der Vermessungskunde, volume IV, second part, pages 1159 to 1164, a method of transformation based largely on Hristow's method. This method was then adopted for use on the *Australian Map Grid* and appropriate tables computed. Formulae, tables and pro forma calculation sheets are given in the *Australian Map Grid Technical Manual*, Special Publication No. 7. However, with the ready availability of modern electronic computers, and other methods of transformation, Hristow's method must now generally be considered obsolete.

9.2 Transformation Based on Bivariate Interpolation into a Table of Complex Numbers

In those cases where vital information relating to either or both projections is not available, for example: the exact unit of measurement, the position of the origin of one system or its orientation relative to the other system, or the exact value of the standard parallel or the central meridian is not known; the problem can in certain circumstances be solved relatively easily. The method requires only that both projections be orthomorphic transformations from the same spheroid and that the rectangular coordinates of several stations common to both systems be known.

The method is based on bivariate interpolation into a table of complex numbers. It was discovered by Professor G.B. Lauf of the University of the Witwatersrand, Johannesburg and was published in *Bulletin Geodesique*, 1961–1962 (pages 191 to 212) under the title 'Conformal Transformations from one Map Projection to another, using Divided Difference Interpolation'.

Suppose that the rectangular coordinates are determined by means of a conformal transformation from any spheroid of reference by two different processes. Let the coordinates on one system be (y, x) and on the other system (Y, X). Then

$$y+ix=F(\psi+i\omega) \quad \text{and} \quad Y+iX=f(\psi+i\omega) \qquad \ldots\ldots 9.1$$

in which ψ is the *isometric latitude* and ω the longitude of the point. Then, from a well known theorem in the theory of functions of a complex variable.

$$Y+iX=f(y+ix) \quad \ldots\ldots 9.2$$

or written in the form of *complex numbers*

$$Z=f(z) \quad \ldots\ldots 9.3$$

Suppose now that several common stations on the two systems are known, then the various sets of coordinates can be set-out in the form of complex numbers together with *divided differences* of various orders, as follows

Station	First System	Second System	Divided Differences		
			First	Second	Third
P_1	$z_1=y_1+ix_1$	$Z_1=Y_1+iX_1$			
			ΔZ_1		
P_2	$z_2=y_2+ix_2$	$Z_2=Y_2+iX_2$		$\Delta^2 Z_1$	
			ΔZ_2		$\Delta^3 Z_1$
P_3	$z_3=y_3+ix_3$	$Z_3=Y_3+iX_3$		$\Delta^2 Z_2$	
			ΔZ_3		
P_4	$z_4=y_4+ix_4$	$Z_4=Y_4+iX_4$			

The various divided differences are defined as follows

First order

$$\Delta Z_1=\frac{Z_2-Z_1}{z_2-z_1}$$

$$\Delta Z_2=\frac{Z_3-Z_2}{z_3-z_2}$$

$$\Delta Z_3=\frac{Z_4-Z_3}{z_4-z_3} \quad \ldots\ldots 9.4$$

Second order

$$\Delta^2 Z_1=\frac{\Delta Z_2-\Delta Z_1}{z_3-z_1}$$

$$\Delta^2 Z_2=\frac{\Delta Z_3-\Delta Z_2}{z_4-z_2} \quad \ldots\ldots 9.5$$

Third order

$$\Delta^3 Z_1=\frac{\Delta^2 Z_2-\Delta^2 Z_1}{z_4-z_1} \quad \ldots\ldots 9.6$$

Suppose now that a station P has coordinates (y, x) on the first system, then we require the coordinates (Y, X) of the same station on the second system. Assume for the moment that the result $Z=Y+iX$, corresponding to $z=y+ix$, is known, then inserting these two values at the head of the original table, we could form divided differences, as follows

$$\Delta Z=\frac{Z_1-Z}{z_1-z}$$

$$\Delta^2 Z=\frac{\Delta Z_1-\Delta Z}{z_2-z}$$

$$\Delta^3 Z=\frac{\Delta^2 Z_2-\Delta^2 Z}{z_3-z} \quad \ldots\ldots 9.7$$

Then, by successive substitutions in equations 9.7, we have

$$Z=Z_1+(z-z_1)\,\Delta Z$$

$$Z=Z_1+(z-z_1)\,\Delta Z_1+(z-z_1)(z-z_2)\,\Delta^2 Z$$

$$Z=Z_1+(z-z_1)\,\Delta Z_1+(z-z_1)(z-z_2)\,\Delta^2 Z_1+(z-z_1)(z-z_2)(z-z_3)\,\Delta^3 Z \quad \ldots\ldots 9.8$$

For *conformal map projections* of the kind normally encountered in surveying, we find that for points not more than a few hundred kilometres apart third order differences are very small and may be considered constant, so that fourth order differences may be neglected. In that case we may substitute $\Delta^3 Z = \Delta^3 Z_1$ in the last of equations 9.8. This gives the interpolation formula

$$Z = Z_1 + (z - z_1)\,\Delta Z_1 + (z - z_1)(z - z_2)\,\Delta^2 Z_1 + (z - z_1)(z - z_2)(z - z_3)\,\Delta^3 Z_1 \qquad \ldots\ldots 9.9$$

As a check on the results, we could interpolate upwards from the bottom of the table and find

$$Z = Z_4 + (z - z_4)\,\Delta Z_3 + (z - z_4)(z - z_3)\,\Delta^2 Z_2 + (z - z_4)(z - z_3)(z - z_2)\,\Delta^3 Z_1 \qquad \ldots\ldots 9.10$$

Theoretically these two results should agree exactly, providing a useful check on the arithmetic involved.

Alternatively, we could interchange systems, using the complex numbers of the second system as argument and those of the first system as entries; and these results again could be checked by interpolating upwards from the bottom of the table.

The various processes involved in this theory of the method are quite valid on the ground that addition, subtraction, multiplication and division of complex numbers can be carried out as for real numbers, provided that $i = \sqrt{-1}$. The algebra of complex numbers in respect of $z_1 = y_1 + ix_1$ and $z_2 = y_2 + ix_2$ takes the following form

Addition $\quad z_1 + z_2 = (y_1 + y_2) + i\,(x_1 + x_2)$

Subtraction $\quad z_1 - z_2 = (y_1 - y_2) + i\,(x_1 - x_2)$

Multiplication $\quad z_1\, z_2 = (y_1\, y_2 - x_1\, x_2) + i\,(y_2\, x_1 + y_1\, x_2)$

Division $\quad z_1 / z_2 = \dfrac{(y_1\, y_2 + x_1\, x_2) + i\,(y_2\, x_1 - y_1\, x_2)}{y_2^2 + x_2^2} \qquad \ldots\ldots 9.11$

Many desk type electronic computers have built-in subroutines to enable the calculations envisaged in equations 9.11 to be carried out automatically, and this of course also applies to the larger electronic computers.

In practice, the method can be applied to two or more common stations taking an appropriate number of terms in the equation of transformation. For small surveys covering no more than a few square kilometres two common stations are sufficient. For surveys up to about 100 kilometre square, three common stations are sufficient. Four common stations are sufficient for surveys up to 10 000 kilometre square. For still larger areas, five common stations could be used, or else, the whole area should be broken down into two or more smaller areas.

In areas where many common stations are available, a *least squares adjustment* can be applied based on a formula equivalent to taking third or fourth order differences. In a particular case, covering about 10 000 kilometre square with more than 100 common stations, it was found that there was no significant improvement in the results when a complex polynomial of degree four or higher was used.

When an electronic computer is used, the calculation can be carried out equally well from system one to system two, or from system two to system one.

Example 9.1

In this example it is assumed that nothing is known about the coordinates of the first system except that the system is an orthomorphic transformation from the same spheroid as the second system. The second system is the Universal Transverse Mercator System with 6° zones and metres as the unit.

Station	First System		UTM System, 6° zone	
	y	x	Y (metre)	X (metre)
A	24 383.284	755 080.095	608 443.84	7 148 122.86
B	23 088.760	756 313.278	613 261.86	7 143 391.43
C	24 264.902	757 873.522	608 730.44	7 137 566.75
D	26 369.462	755 906.865	600 895.80	7 145 114.60

First Order Divided Differences

−3.776 5175	+0.057 3901
−3.776 5208	+0.057 4879
−3.776 4050	+0.057 4736

Second Order Divided Differences

$+0.035\ 00\times10^{-6}$	$-0.000\ 30\times10^{-6}$
$+0.035\ 30\times10^{-6}$	$+0.000\ 01\times10^{-6}$

Third Order Divided Differences

$+0.184\times10^{-12}$ $\quad +0.079\times10^{-12}$

Point to be transformed P : y = +24 719.441 x = +756 286.865 (First System)

Transformation to UTM System

608 443.84	7 148 122.86
−1 338.759	−4 538.096
+0.021	+0.068
+0.001	+0.000
P: 607 105.102	7 143 584.833 (UTM System)

Transformation from UTM System to First System

Point to be transformed P : y = +607 105.102 x = +7 143 584.833 (UTM System)

24 383.284	755 080.095
+336.152	+1 206.752
+0.005	+0.018
+0.000	+0.000
P: 24 719.441	756 286.865 (First System)

Transformation to UTM System, interpolating from the bottom of the table upwards

Point to be transformed P : y = +24 719.441 x = +756 286.865 (First System)

600 895.80	7 145 114.60
+6 209.308	−1 529.867
−0.005	+0.099
−0.000	+0.001
P: 607 105.102	7 143 584.833 (UTM System)

From the values given in the First Order Divided Differences Table, it is clear that the unit of measurement used in the First System is approximately 3.776 metre.

Example 9.2

In this example, we transform from System I, which is on Mercator's projection in metres, to System II, which is on the Stereographic projection (equatorial case) in metres. The area covered is about 90 000 kilometre square, that is about 300×300 kilometre. The transformation is based on five common stations.

	System I (Mercator's Projection)		System II (Stereographic Projection)	
Station	Easting (metre)	Northing (metre)	Easting (metre)	Northing (metre)
P_1	3 504 035.296	3 040 363.285	3 605 191.375	2 864 164.673
P_2	3 391 781.487	3 193 570.660	3 515 491.656	3 028 102.576
P_3	3 211 911.472	3 134 882.351	3 331 113.806	2 993 211.802
P_4	3 211 911.472	2 945 844.219	3 308 694.968	2 807 209.252
P_5	3 391 781.487	2 887 155.910	3 477 506.847	2 729 105.397

First Order Divided Differences

0.975 389 454	−0.129 180 873
0.983 635 391	−0.126 964 596
0.983 941 959	−0.118 594 263
0.976 267 600	−0.115 685 543

Second Order Divided Differences

$-0.233\ 303\ 262 \times 10^{-7}$	$-0.151\ 354\ 930 \times 10^{-7}$
$-0.227\ 129\ 593 \times 10^{-7}$	$-0.152\ 539\ 715 \times 10^{-7}$
$-0.224\ 169\ 634 \times 10^{-7}$	$-0.147\ 025\ 870 \times 10^{-7}$

Third Order Divided Differences

$-0.179\ 430\ 110 \times 10^{-14}$	$0.986\ 137\ 426 \times 10^{-15}$
$-0.179\ 947\ 133 \times 10^{-14}$	$0.965\ 997\ 584 \times 10^{-15}$

Fourth Order Divided Difference

$0.101\ 624\ 700 \times 10^{-21}$	$0.407\ 129\ 968 \times 10^{-22}$

Coordinates of station P to be transformed from System I to System II

P : y = +3 341 651.967   x = +3 040 363.285 (System I)

The coordinates of station P on System II are

3 605 191.375	2 864 164.673	
−158 386.987	20 976.820	
186.632	−703.625	
−8.538	−1.051	
−0.005	0.074	
P: 3 446 982.477	2 884 436.891	(System II)

The transformation back from System II to System I gives

3 504 035.296	3 040 363.285	
−162 109.871	−686.164	
−278.952	678.146	
5.552	7.913	
−0.057	0.105	
P: 3 341 651.968	3 040 363.286	(System I)

This agrees with the original coordinates of P to ±0.001 metre in each case.

The transformation to System II, interpolating from the bottom of the table upwards, gives

3 477 506.847	2 729 105.397	
−31 215.948	+155 370.657	
+692.999	−30.839	
−1.489	−8.351	
+0.068	0.027	
P: 3 446 982.477	2 884 436.891	(System II)

Example 9.3

In this example, a computer programme has been written to enable coordinates to be transformed from AMG zone 54 to zone 55, or from zone 55 to zone 54, for any point in the State of Victoria. The transformation is based on five selected points in the common overlap, as follows

	AMG — Zone 54		AMG — Zone 55	
	E	N	E	N
A	766 963.090	5 900 905.064	233 036.910	5 900 905.064
B	725 342.160	6 013 146.972	184 491.051	6 010 369.313
C	815 508.949	6 010 369.313	274 657.840	6 013 146.972
D	807 338.419	5 788 387.009	280 488.865	5 791 220.908
E	719 511.135	5 791 220.908	192 661.581	5 788 387.009

Station to be transformed, P (zone 54)

P : y = +729 627.998   x = +5 857 987.855 (zone 54)

Coordinates of P on zone 55

233 036.910	5 900 905.064
−34 599.753	−45 172.880
48.478	−32.826
−0.133	+0.389
±0.000	+0.001
P: 198 485.501	5 855 699.747 (zone 55)

Station to be transformed, P (zone 55)

P : y = +198 485.501 x = +5 855 699.747 (zone 55)

Coordinates of P on zone 54

766 963.090	5 900 905.064
−37 288.875	−42 952.584
−46.448	−35.714
+0.231	−0.341
±0.000	+0.001
P: 729 627.999	5 857 987.855 (zone 54)