Chapter 10

*Analytical Solution of the Problem of Topographic Mapping**

58. Comparator Measurements. This involves finding the space coordinates or the two horizontal rectangular coordinates on the ground and the elevation above a datum plane. This is done from the measurements made on a photograph, and the measurements are made on the comparator, which reads to 0.01 mm.

The conditions which must be fulfilled in the field work are two. (1) Every point must appear in two photographs. (2) Each photograph must contain three ground-control points. The first of these conditions is fulfilled by the usual overlap. The second can be fulfilled by selecting three well-defined points in the photograph and surveying these points in the field for both horizontal and vertical control.

In order to solve the complete problem, it is necessary to compute the following initial data: (1) the elements of interior orientation for the aerial camera used; (2) the space coordinates on the ground or survey system of the exposure station of each photograph; (3) the elements of exterior orientation for each photograph.

59. Elements of Interior and Exterior Orientation Explained. The elements of interior orientation for a particular aerial camera are three, the two rectangular coordinates on the plate of the *principal point*, or the point of intersection of the camera axis with the plane of the plate, and the *principal distance* of the camera. The coordinates on the plate of the principal point differ very slightly from zero, for the principal point very nearly coincides with the collimating point photographed on the plate, which is used as the origin of the plate system of coordinates. The principal distance of the camera, commonly but erroneously called the focal length, is the distance from the plane of the film to the nodal point of the lens, the point through which all rays of light are assumed to pass in a straight line from object to image, the vertex of the conical projection. In the procedure with this discussion, it will be assumed that the plate coordinates of the principal point are O, O, and that the camera principal distance, designated by f, is known.

The exposure station, which is exactly the nodal point of the lens mentioned above, is designated by L. Its space coordinates x_L, y_L, z_L, on the ground or survey system, must be accurately determined, and the method of computing them will be explained here.

The elements of exterior orientation are three. The first, called the *swing*, is the angle on the plate between the positive direction of the plate Y-axis and the *principal line*, this angle being exactly analogous to what is called in surveying the *azimuth* on the plate system of the principal line. The principal line is the line on the plate joining the principal point, o, to the *nadir point*, v, the nadir point in turn being the point where the plane of the plate is intersected by a vertical line through the

* The analytical system was worked out by Professor Earl Church at Syracuse University, and acknowledgment is here made for his assistance in writing this book.

exposure station. The swing will be designated by s.

The second element is the tilt, or the angle between the axis of the camera lens and a vertical line, or the angle between the plane of the plate and a horizontal plane. The tilt will be designated by t. The third element is the azimuth of the *principal plane* on the ground or survey system, the principal plane being the vertical plane through the exposure station including the axis of the camera. The symbol for this element will be Az_{vo}, indicating the survey azimuth of the line from V, the point on the ground vertically beneath L, to O, the point where the ground is intersected by the axis of the camera. The determination of these elements of exterior orientation will likewise be completely explained later.

After the elements of interior orientation have been found for the aerial camera and the coordinates of the exposure station and the elements of exterior orientation have been computed for each photograph, it will remain to show how the plate coordinates of the image of any point determine the survey coordinates of that point.

60. Determination of the Space Coordinates of the Exposure Station. The various steps in the determination of the coordinates on the ground or survey system of the exposure station, as they would be executed for each photograph, are given here in their proper order.

1. A tabulation is made of the control data, that is, of the ground coordinates of A, B, and C, the three control points, these coordinates being designated by (x_A, y_A, z_A), (x_B, y_B, z_B), and (x_C, y_C, z_C).

2. The plate positive is next placed in the comparator and precise measurements are made of the plate coordinates of the images, a, b, and c, of the three control points. As mentioned before, the origin of this plate system of plane coordinates is the principal point, o, and the axes are the geometric axes of the plate, all of which are indicated by marks photographed on the original film simultaneously with the field exposure. The Y-axis will be taken in the direction of flight. The measured coordinates of the three images on the plate system will be designated by $(x_a y_a)$ $(x_b y_b)$ $(x_c y_c)$.

3. The next step is to compute the distances from the nodal point or exposure station, L, to each of the three images, a, b, and c. Using Fig. 78, we may write the following equations by the principles of analytics:

$$\left.\begin{aligned} La &= \sqrt{x_a^2 + y_a^2 + f^2} \\ Lb &= \sqrt{x_b^2 + y_b^2 + f^2} \\ Lc &= \sqrt{x_c^2 + y_c^2 + f^2} \end{aligned}\right\} \quad [3]$$

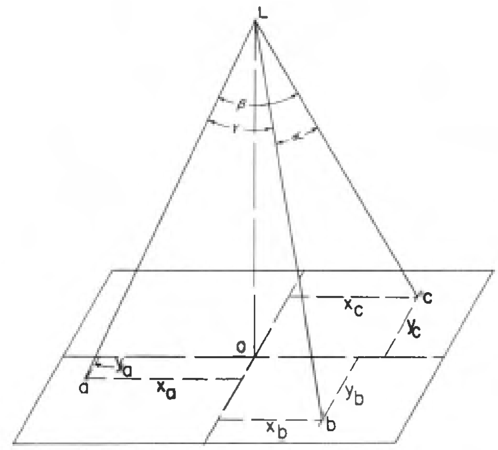

Fig. 78.

4. On this same plate system of space coordinates, using the usual formulas of analytics for the direction cosines of a line joining two points, we find that the direction cosines of La, Lb, and Lc are:

$$\text{For } La, \frac{x_a}{La}, \frac{y_a}{La}, \frac{-f}{La}; \quad \text{for } Lb, \frac{x_b}{Lb}, \frac{y_b}{Lb}, \frac{-f}{Lb};$$

$$\text{for } Lc, \frac{x_c}{Lc}, \frac{y_c}{Lc}, \frac{-f}{Lc}$$

Then the cosine of the angle between two lines gives directly the cosines of the angles aLb, bLc, and cLa, the face angles at the apex L of the plate pyramid L–abc, designated respectively by gamma, alpha, and beta. The formulas are:

From analytics:

$$\cos\gamma = \frac{x'x'' + y'y'' + z'z''}{r'r''}$$

Then:

$$\left.\begin{aligned} \cos\gamma &= \frac{x_a x_b + y_b y_a + f^2}{La \times Lb} \\ \cos\alpha &= \frac{x_b x_c + y_b y_c + f^2}{Lb \times Lc} \\ \cos\beta &= \frac{x_c x_a + y_c y_a + f^2}{Lc \times La} \end{aligned}\right\} \quad [4]$$

5. It now becomes necessary before proceeding with the computation to find approximate values for the required space coordinates of L on the ground or survey system. When the tilt is not large it will be satisfactory to obtain these approximate values as follows: Plot A, B, and C to scale from the control data; from the plate trace rays on tracing paper from the principal point o to the images of a, b, and c; transfer to the plot a point corresponding to o, by the regular process of solving the three-point problem graphically; scale from the plot

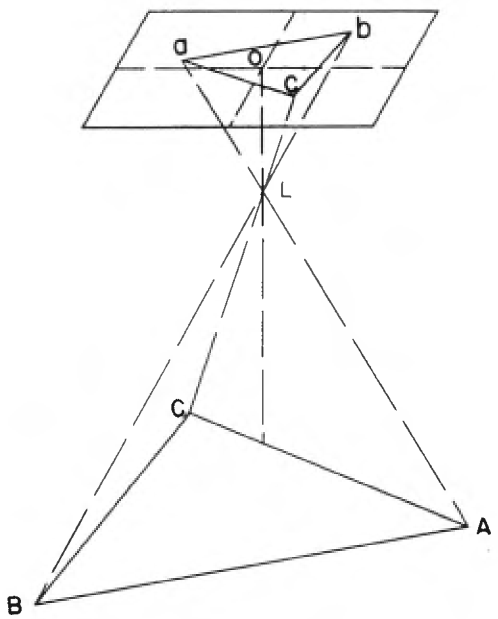

FIG. 79.

the coordinates of this point, calling the scaled distance the approximate X and Y coordinates of L; scale the distance AB, say, from the plot and measure the corresponding distance ab on the plate; find H, the approximate height of flight above A and B, from the simple scale proportion $H/f = AB/ab$; add this value of H to the average of the elevations of A and B, taking this result as the approximate Z coordinate of the exposure station.

Call the approximate values for the space coordinates on the ground or survey system of the point L, (x_L), (y_L), (z_L). The true coordinates will be represented by the same value without the parentheses, namely, x_L, y_L, z_L.

6. The next step is to compute the corresponding approximate values of the lengths of the edges LA, LB, and LC, of the ground pyramid L–ABC. Using parentheses again to indicate approximate values corresponding to our approximate coordinates of L, and using again the well-known analytics formula for the distance between two points, we have the relations

$$\left.\begin{aligned}(LA) &= \sqrt{[(x_L)-x_A]^2+[(y_L)-y_A]^2+[(z_L)-z_A]^2}\\(LB) &= \sqrt{[(x_L)-x_B]^2+[(y_L)-y_B]^2+[(z_L)-z_B]^2}\\(LC) &= \sqrt{[(x_L)-x_C]^2+[(y_L)-y_C]^2+[(z_L)-z_C]^2}\end{aligned}\right\} \quad [5]$$

(See Fig. 79.)

7. Inasmuch as the direction cosines of the lines LA, LB, and LC are respectively:

$$\frac{x_L - x_A}{LA}, \quad \frac{y_L - y_A}{LA}, \quad \frac{z_L - z_A}{LA}$$

and

$$\frac{x_L - x_B}{LB}, \quad \frac{y_L - y_B}{LB}, \quad \frac{z_L - z_B}{LB}$$

and

$$\frac{x_L - x_C}{LC}, \quad \frac{y_L - y_C}{LC}, \quad \frac{z_L - z_C}{LC}$$

and, as the apex angles at L in the ground pyramid L–ABC are exactly equal to the apex angles at L in the plate pyramid L–abc, it follows from the usual analytics formula for the cosine of the angle between two lines in space that

$$\left.\begin{aligned}\cos\gamma &= \frac{(x_L-x_A)(x_L-x_B)+(y_L-y_A)(y_L-y_B)+(z_L-z_A)(z_L-z_B)}{LA \times LB}\\\cos\alpha &= \frac{(x_L-x_B)(x_L-x_C)+(y_L-y_B)(y_L-y_C)+(z_L-z_B)(z_L-z_C)}{LB \times LC}\\\cos\beta &= \frac{(x_L-x_C)(x_L-x_A)+(y_L-y_C)(y_L-y_A)+(z_L-z_C)(z_L-z_A)}{LC \times LA}\end{aligned}\right\} \quad [6]$$

NOTE: The above formulas come from the formula in analytics, that the cosine of the angle between two lines is

$$\cos\gamma = \frac{x'x'' + y'y'' + z'z''}{r'r''}$$

The next step in the computation is to find the amounts by which the approximate coordinates of L fail to satisfy these relations. It might be noted that if the approximate coordinates of L satisfy these equations exactly, the approximate values would be exactly equal to the actual values of the coordinates of L. Using v's to designate the residuals obtained when the approximate values of the coordinates of L are substituted in these equations cleared of fractions, we have

$$\left.\begin{aligned} v_1 &= [(x_L)-x_A][(x_L)-x_B]+[(y_L)-y_A][(y_L)-y_B] \\ &\quad +[(z_L)-z_A][(z_L)-z_B]-(LA)(LB)\cos\gamma \\ v_2 &= [(x_L)-x_B][(x_L)-x_C]+[(y_L)-y_B][(y_L)-y_C] \\ &\quad +[(z_L)-z_B][(z_L)-z_C]-(LB)(LC)\cos\alpha \\ v_3 &= [(x_L)-x_C][(x_L)-x_A]+[(y_L)-y_C][(y_L)-y_A] \\ &\quad +[(z_L)-z_C][(z_L)-z_A]-(LC)(LA)\cos\beta \end{aligned}\right\} \quad [7]$$

8. We are now concerned with finding what corrections are necessary to the approximate coordinates of L to obtain values which will exactly satisfy the relations of the last paragraph and thereby make the apex angles of the ground pyramid L–ABC exactly equal, as they should, the apex angles of the plate pyramid L–abc.

The changes which the desired corrections to (x_L), (y_L), and (z_L) are to produce in the right-hand sides of the last equations must be equal to $-v_1$, $-v_2$, $-v_3$.

Let us put for (LA), (LB), (LC) the expressions of paragraph 6 in terms of coordinates, and find by differentiation the changes in the right-hand sides of the last relations in paragraph 7 produced by changes in (x_L), (y_L), (z_L). Thus

$$\begin{aligned} v_1 = {} & [(x_L)-x_A][(x_L)-x_B]+[(y_L)-y_A][(y_L)-y_B]+[(z_L)-z_A][(z_L)-z_B] \\ & -\sqrt{[(x_L)-x_A]^2+[(y_L)-y_A]^2+[(z_L)-z_A]^2} \\ & \quad \sqrt{[(x_L)-x_B]^2+[(y_L)-y_B]^2+[(z_L)-z_B]^2}\times\cos\gamma \end{aligned}$$

$$\begin{aligned} -v_1 = {} & [2(x_L)-x_A-x_B]\Delta x+[2(y_L)-y_A-y_B]\Delta y+[2(z_L)-z_A-z_B]\Delta z \\ & -\cos\gamma\,(LA)\frac{[(x_L)-x_B]\Delta x+[(y_L)-y_B]\Delta y+[(z_L)-z_B]\Delta z}{(LB)} \\ & -\cos\gamma\,(LB)\frac{[(x_L)-x_A]\Delta x+[(y_L)-y_A]\Delta y+[(z_L)-z_A]\Delta z}{(LA)} \end{aligned}$$

The above equations can best be handled in the following form:

$$\left.\begin{aligned} &\begin{bmatrix} \left(1-\frac{(LB)}{(LA)}\cos\gamma\right)[(x_L)-x_A] \\ +\left(1-\frac{(LA)}{(LB)}\cos\gamma\right)[(x_L)-x_B] \end{bmatrix}\Delta x+\begin{bmatrix} \left(1-\frac{(LB)}{(LA)}\cos\gamma\right)[(y_L)-y_A] \\ +\left(1-\frac{(LA)}{(LB)}\cos\gamma\right)[(y_L)-y_B] \end{bmatrix}\Delta y+\begin{bmatrix} \left(1-\frac{(LB)}{(LA)}\cos\gamma\right)[(z_L)-z_A] \\ +\left(1-\frac{(LA)}{(LB)}\cos\gamma\right)[(z_L)-z_B] \end{bmatrix}\Delta z \\ &\qquad +v_1=0 \\ &\begin{bmatrix} \left(1-\frac{(LC)}{(LB)}\cos\alpha\right)[(x_L)-x_B] \\ +\left(1-\frac{(LB)}{(LC)}\cos\alpha\right)[(x_L)-x_C] \end{bmatrix}\Delta x+\begin{bmatrix} \left(1-\frac{(LC)}{(LB)}\cos\alpha\right)[(y_L)-y_B] \\ +\left(1-\frac{(LB)}{(LC)}\cos\alpha\right)[(y_L)-y_C] \end{bmatrix}\Delta y+\begin{bmatrix} \left(1-\frac{(LC)}{(LB)}\cos\alpha\right)[(z_L)-z_B] \\ +\left(1-\frac{(LB)}{(LC)}\cos\alpha\right)[(z_L)-z_C] \end{bmatrix}\Delta z \\ &\qquad +v_2=0 \\ &\begin{bmatrix} \left(1-\frac{(LA)}{(LC)}\cos\beta\right)[(x_L)-x_C] \\ +\left(1-\frac{(LC)}{(LA)}\cos\beta\right)[(x_L)-x_A] \end{bmatrix}\Delta x+\begin{bmatrix} \left(1-\frac{(LA)}{(LC)}\cos\beta\right)[(y_L)-y_C] \\ +\left(1-\frac{(LC)}{(LA)}\cos\beta\right)[(y_L)-y_A] \end{bmatrix}\Delta y+\begin{bmatrix} \left(1-\frac{(LA)}{(LC)}\cos\beta\right)[(z_L)-z_C] \\ +\left(1-\frac{(LC)}{(LA)}\cos\beta\right)[(z_L)-z_A] \end{bmatrix}\Delta z \\ &\qquad +v_3=0 \end{aligned}\right\} \quad [8]$$

The constant coefficients in these equations look rather cumbersome but as a matter of fact they are readily computed from values already obtained. The three resulting simultaneous equations in Δx, Δy, and Δz are linear, and can be solved for the corrections Δx, Δy, and Δz.

9. The exact coordinates of L in space can then be found by applying the corrections just found to the approximate values.

$$x_L = (x_L) + \Delta x \qquad y_L = (y_L) + \Delta y$$

$$z_L = (z_L) + \Delta z$$

10. After completing the computations of the exact space coordinates on the ground or survey system of the exposure station L, before proceeding to the computation of the elements of exterior orientation, it is necessary to find the exact lengths of LA, LB, and LC. These are found in the same way as the approximate values.

$$\left.\begin{aligned} LA &= \sqrt{(x_L - x_A)^2 + (y_L - y_A)^2 + (z_L - z_A)^2} \\ LB &= \sqrt{(x_L - x_B)^2 + (y_L - y_B)^2 + (z_L - z_B)^2} \\ LC &= \sqrt{(x_L - x_C)^2 + (y_L - y_C)^2 + (z_L - z_C)^2} \end{aligned}\right\} [9]$$

11. It would be well at this point to substitute the final values of x_L, y_L, z_L, LA, LB, and LC, in the last equations of paragraph 7 to ascertain whether by any chance residuals still remain. If there are still residuals, the process given in paragraphs 7, 8, 9, 10, and 11 should be repeated.

Equation 8 may be set up in the following form when U, V, and W are the coefficients respectively of Δx, Δy, and Δz in the first equation of 8; U', V', and W' are the coefficients of Δx, Δy, and Δz in the second equation; and U'', V'', and W'' are the coefficients of Δx, Δy, and Δz in the third equation.

$$\left.\begin{aligned} U\ \Delta X + V\ \Delta Y + W\ \Delta Z + v_1 &= 0 \\ U'\ \Delta X + V'\ \Delta Y + W'\ \Delta Z + v_2 &= 0 \\ U''\Delta X + V''\Delta Y + W''\Delta Z + v_3 &= 0 \end{aligned}\right| \quad [8a]$$

The coefficients are given by the following expressions:

$$\left.\begin{aligned} U &= [1 - (LA/LB)\cos\gamma](X - X_B) + [1 - (LB/LA)\cos\gamma](X - X_A) \\ V &= [1 - (LA/LB)\cos\gamma](Y - Y_B) + [1 - (LB/LA)\cos\gamma](Y - Y_A) \\ W &= [1 - (LA/LB)\cos\gamma](Z - Z_B) + [1 - (LB/LA)\cos\gamma](Z - Z_A) \\ U' &= [1 - (LB/LC)\cos\alpha](X - X_C) + [1 - (LC/LB)\cos\alpha](X - X_B) \\ V' &= [1 - (LB/LC)\cos\alpha](Y - Y_C) + [1 - (LC/LB)\cos\alpha](Y - Y_B) \\ W' &= [1 - (LB/LC)\cos\alpha](Z - Z_C) + [1 - (LC/LB)\cos\alpha](Z - Z_B) \\ U'' &= [1 - (LC/LA)\cos\beta](X - X_A) + [1 - (LA/LC)\cos\beta](X - X_C) \\ V'' &= [1 - (LC/LA)\cos\beta](Y - Y_A) + [1 - (LA/LC)\cos\beta](Y - Y_C) \\ W'' &= [1 - (LC/LA)\cos\beta](Z - Z_A) + [1 - (LA/LC)\cos\beta](Z - Z_C) \end{aligned}\right\} \quad [8b]$$

The coefficients of ΔX, ΔY, and ΔZ, are computed in 8*b*, using the approximate values of the coordinates of the exposure station.

The subscript L and the brackets have been omitted for simplicity. But X, Y, Z in equations 8*b* have the same meaning as (X_L) (Y_L) (Z_L) in equation 8.

The following forms and computation are printed here with the permission of Professor Earl Church. They were submitted by him to *Photogrammetric Engineering* and published in Volume VII, No. 4, in 1941.

SPACE RESECTION

Photograph No. f				a x_a y_a b x_b y_b c x_c y_c			La [1] Lb Lc	cosine γ [2] cosine α cosine β
L	(X)	(Y)	(Z)					
A	X_A	Y_A	Z_A	$(X)-X_A$	$(Y)-Y_A$	$(Z)-Z_A$	(LA) [3]	$\frac{LB}{LA}\cos\gamma$ $\frac{LC}{LA}\cos\beta$
B	X_B	Y_B	Z_B	$(X)-X_B$	$(Y)-Y_B$	$(Z)-Z_B$	(LB)	$\frac{LA}{LB}\cos\gamma$ $\frac{LC}{LB}\cos\alpha$
C	X_C	Y_C	Z_C	$(X)-X_C$	$(Y)-Y_C$	$(Z)-Z_C$	(LC)	$\frac{LB}{LC}\cos\alpha$ $\frac{LA}{LC}\cos\beta$

1st 3 terms 4th term [4]	1st terms of U, V, W 2nd terms of U, V, W [5]	U X U' X	V Y V' Y	W Z W' Z	v_1 v_2		
v_1 1st 3 terms	1st terms of U', V', W' 2nd terms of U', V', W' [5]	U'' X	V'' Y	W'' Z	v_3		
4th term [4]	1st terms of U'', Y'', W''						
v_2	2nd terms of U'', V'', W'' [5]	Solutions of equations					
1st 3 terms	(X)	(Y)	(Z)				
4th term [4]	$\frac{\Delta X}{X}$	$\frac{\Delta Y}{Y}$	$\frac{\Delta Z}{Z}$				
v_3							

Second Solution

	(X) $\frac{\Delta X}{X}$	(Y) $\frac{\Delta Y}{Y}$	(Z) $\frac{\Delta Z}{Z}$
	Exposure station		

$X-X_A$ $X-X_B$ $X-X_C$	$Y-Y_A$ $Y-Y_B$ $Y-Y_C$	$Z-Z_C$ $Z-Z_B$ $Z-Z_C$	LA [3] LB LC	1st 3 terms 4th term [4]	1st 3 terms 4th term [4]	1st 3 terms 4th term [4]
				v_1	v_2	v_3

[1] See Eq. 3.
[2] See Eq. 4.
[3] See Eq. 5.
[4] From Eq. 7.
[5] From Eq. 8b.

Space Resection

Photograph I
f 150.00 mm.

a	3.68	−71.56	166.236	.907084
b	82.29	−74.88	186.758	.648273
c	83.56	83.56	190.957	.530115

L	4600	34500	19785							
A	5000	25000	400	− 400	9500	19385	21591.39	.986		.591
B	15000	25000	1000	−10400	9500	18785	23479.49	.834	.664	
C	15000	45000	800	−10400	−10500	18985	24059.10		.633	.476

458557225	− 6	133	271	−7311	− 662	13279	−1163209
−459850611	−1726	1577	3118	−5614	−1616	17876	−2944302
− 1293386	−3494	3192	6312	−1732	1710	3389	−1293386
365043225	−3817	−3854	6967	−1866	− 169	3389	− 296868
−366206434				−1064	− 306	3389	− 558192
	− 164	3886	7928				
− 1163209	−5450	−5502	9948	134	1879		− 996518
272434225				802	− 137		− 261324
−275378527	4600	34500	19785				
	411	501	339	10	137		− 72657
− 2944302							
	5011	35001	20124	812			− 333981

Second Solution

L	5011	35001	20124
A	5000	25000	400
B	15000	25000	1000
C	15000	45000	800

11	10001	19724	22114.62	.975		.574
− 9989	10001	19124	23780.82	.844	.653	
− 9989	− 9999	19324	23941.12		.644	.490

477111898	0	250	493	−7022	− 90	13515	244895
−477038920	−1558	1560	2983	−5089	− 839	18257	367963
72978	−3466	3470	6636	−1558	1810	3476	72978
369332298	−3556	−3560	6879	−1806	− 23	3476	62986
−369087403				− 969	− 160	3476	70057
	5	4260	8402				
244895	−5094	−5099	9855	248	1833		9992
281036698				837	− 137		7071
−280668735	5011	35001	20124				
	− 9	− 4	− 23	19	137		747
367963							
	5002	34997	20101	856			7818
	Exposure station						

2	9997	19701	22092.29	476228814	368628414	280229014
−9998	9997	19101	23764.43	−476228787	−368630745	−280231562
−9998	−10003	19301	23927.99			
				27	− 2331	− 2548

61. Elements of Exterior Orientation.

(*a*) Tilt.
(*b*) Survey azimuth of the principal plane.
(*c*) Swing.

With the space coordinates of the exposure station determined, the spatial orientation of the aerial photograph may be expressed in several different ways. It is customary to state this so-called exterior orientation as three distinct elements which will be defined.

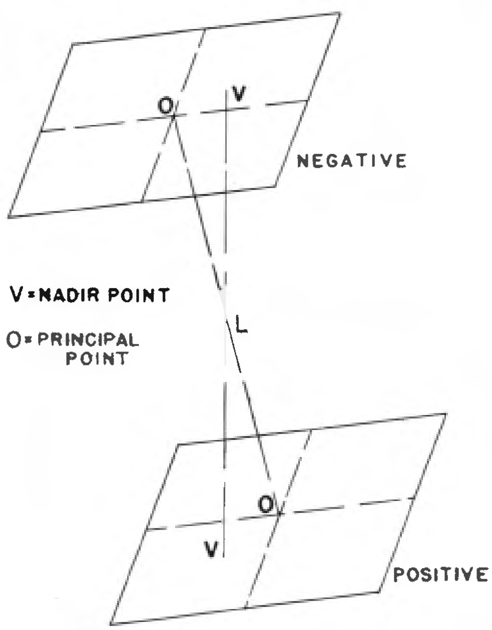

FIG. 80.

The first element is the *tilt*, the angle between a horizontal plane and the plane of the photographic film at the instant of exposure or, what is the same thing, the angle between the camera axis and a vertical line.

In Fig. 80 there is shown a negative made from the exposure station *L*. Together with this is shown the corresponding diapositive. Inasmuch as all our measurements are made upon contact positive prints on glass plates, it is convenient to conceive the definitions of the elements of exterior orientation as they apply to a plate positive. Subsequent figures will, therefore, be drawn like Fig. 81.

In Fig. 81 only the diapositive is shown without the negative. The angle *OLV* is the tilt, the first element of exterior orientation. The point *v*, where the plane of the photograph is intersected by a vertical line through the exposure station or emergent nodal point of the camera lens, is called the nadir point. The line *OV* on the photographic plate joining the principal point and the nadir point is called the principal line. The vertical plane *OLV* containing the camera axis is called the principal plane. The principal line indicates the direction of the tilt, and lines on the photograph perpendicular to the principal line are horizontal.

The second element of exterior orientation is the ground survey azimuth of the principal plane. It is the survey azimuth of *VO*, and is designated by *Az*. Like all other azimuths in photographic surveying, it is usually measured clockwise from north.

The third element of exterior orientation is called the swing. This is the photographic direction of the principal line *OV*. It is taken as the angle on the photograph from the positive direction of the *Y*-axis, clockwise to the principal line *OV*, in the same manner as an azimuth. The swing is shown in Fig. 81 as the angle *YOV*, designated by *s*.

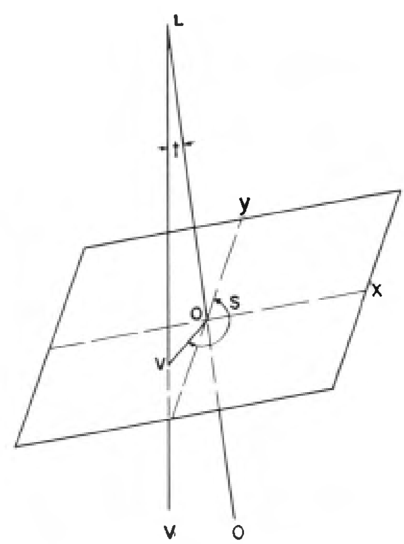

FIG. 81.

The problem is now to show how to calculate these three elements of exterior orientation of the photograph from the comparator measurements of the photographic rectangular coordinates of the images of the three control points, the same measurements in fact which were used in the previous articles.

In Fig. 82 a, b, c represent the images of the three control points A, B, C. The lengths La, Lb, and Lc have already been computed. If we use again a temporary system of coordinates with the origin at L, X- and Y-axes parallel respectively to the

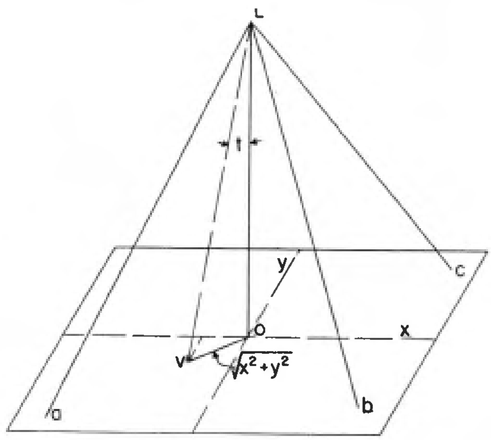

FIG. 82.

axes of the plate, and Lo for the Z-axis, the *known coordinates of the images* a, b, and c are x_a, y_a, $-f$; x_b, y_b, $-f$; x_c, y_c, $-f$. The *direction cosines* of La, Lb, and Lc are

$$\frac{x_a}{La}, \frac{y_a}{La}, \frac{-f}{La}; \quad \frac{x_b}{Lb}, \frac{y_b}{Lb}, \frac{-f}{Lb}; \quad \frac{x_c}{Lc}, \frac{y_c}{Lc}, \frac{-f}{Lc}.$$

Use x and y to designate the unknown photographic plane coordinates of the nadir point v. Then on the space coordinate system being used, the coordinates of v are x, y, $-f$; the length of Lv is expressed by

$$\sqrt{x^2 + y^2 + f^2}$$

and the direction cosines of Lv are expressed by

$$\frac{x}{\sqrt{x^2+y^2+f^2}}; \quad \frac{y}{\sqrt{x^2+y^2+f^2}}; \quad \frac{-f}{\sqrt{x^2+y^2+f^2}}$$

The unknown angles vLa, vLb, and vLc are designated by m_a, m_b, and m_c (Fig. 83). By following the usual method of space analytic geometry for finding the angle between two lines, these angles (in terms of the unknown quantities x and y) are expressed by the following relations:

$$\left.\begin{aligned} \cos m_a &= \frac{x_a x + y_a y + f^2}{La\sqrt{x^2+y^2+f^2}} \\ \cos m_b &= \frac{x_b x + y_b y + f^2}{Lb\sqrt{x^2+y^2+f^2}} \\ \cos m_c &= \frac{x_c x + y_c y + f^2}{Lc\sqrt{x^2+y^2+f^2}} \end{aligned}\right\} \qquad [10]$$

Inasmuch as the space coordinates of L, A, B, and C with respect to the survey axes are all known, it is possible to calculate the cosines of the angles between the line LV and each of the lines LA, LB, and LC. (See Fig. 83.)

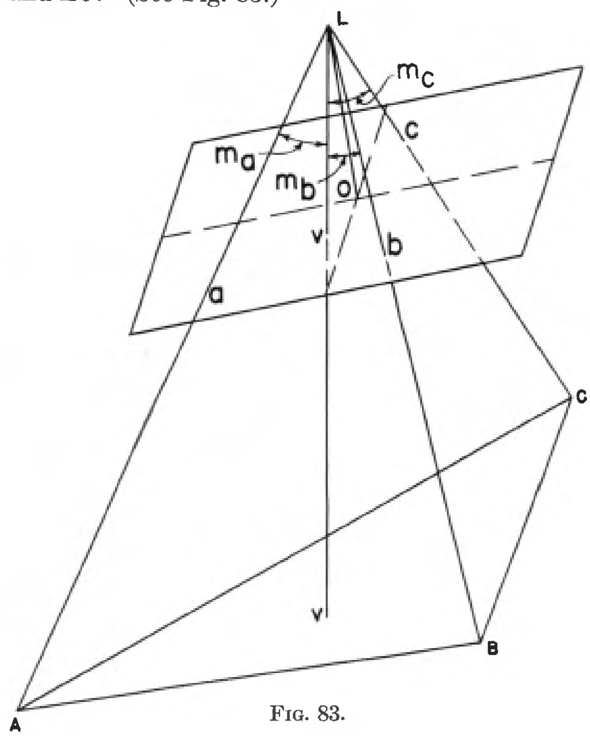

FIG. 83.

If we use L_H to designate the projection of L upon a horizontal plane through A, as shown in Fig. 84, the following relations exist:

$$\cos m_A = \frac{z_L - z_A}{LA}$$

$$\cos m_B = \frac{z_L - z_B}{LB}$$

$$\cos m_C = \frac{z_L - z_C}{LC}$$

The final values of LA, LB, and LC have already been calculated in the preceding discussion.

To find the plate coordinates x, y of the nadir point v, the problem resolves itself into finding values for these quantities which will satisfy the equations in the preceding column when the known numerical values of the cosines are used therein. These equations constitute three simultaneous equations with but two unknowns (x and y). Two of these

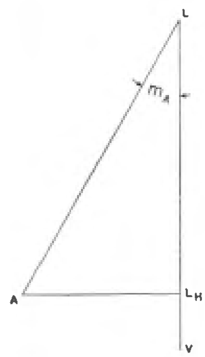

FIG. 84.

equations could be used to solve for x and y, and if the exposure station has been correctly determined the values obtained will satisfy the third equation. However, the solution would be difficult, for the equations would be complicated quadratics. A simpler way to find x and y is to combine the equations in the preceding column algebraically to obtain two linear equations. This transformation is as follows since $m_a = m_A$, $m_b = m_B$, $m_c = m_C$.

$$\left.\begin{aligned} \sqrt{x^2+y^2+f^2} &= \frac{x_a x + y_a y + f^2}{La \cos m_A} \\ \sqrt{x^2+y^2+f^2} &= \frac{x_b x + y_b y + f^2}{Lb \cos m_B} \\ \sqrt{x^2+y^2+f^2} &= \frac{x_c x + y_c y + f^2}{Lc \cos m_C} \end{aligned}\right\} \quad [11]$$

Then, from Eqs. 11,

$$\frac{x_a x + y_a y + f^2}{La \cos m_A} = \frac{x_b x + y_b y + f^2}{Lb \cos m_B} = \frac{x_c x + y_c y + f^2}{Lc \cos m_C}$$

Transposing and forming the following equation,* we have

$$\left.\begin{aligned} &\left(\frac{x_a}{La \cos m_A} - \frac{x_b}{Lb \cos m_B}\right)x \\ &+ \left(\frac{y_a}{La \cos m_A} - \frac{y_b}{Lb \cos m_B}\right)y \\ &+ \left(\frac{f^2}{La \cos m_A} - \frac{f^2}{Lb \cos m_B}\right) = 0 \end{aligned}\right\} \quad [12]$$

The coefficients of Eqs. 12 can be computed, and the solution of this pair of linear equations for x and y gives the plate coordinates of the nadir point v.

Then from Fig. 82, $ov = \sqrt{x^2 + y^2}$, and the angle oLv, which is the *tilt*, one of the desired elements of exterior orientation, can be found directly from Eq. 13.

$$\tan t = \frac{\sqrt{x^2 + y^2}}{f} \quad [13]$$

The *swing*, another element of exterior orientation, is the angle *yov* or the plate azimuth of *ov*. It is found from the relation in Eq. 14.

$$\tan s = \frac{x}{y} \quad [14]$$

The proper quadrant for s is determined from the plate coordinates of v.

Having completed the determination of two of the three elements of exterior orientation, we have yet to find the third. In the work which follows it is convenient to have the nadir point v and the principal line precisely marked upon each plate. This can be accurately done by a device on the comparator. The principal line should be indicated accurately by marking another point u near the margin of the photograph on *ov* produced (Fig. 85). One coordinate of u may be assumed arbitrarily so that the point will fall near the margin of the plate, and the other coordinate can be found from Eq. 15.

$$\tan s = \frac{x_u}{y_u} \quad [15]$$

The point u can be marked from its coordinates.

* A similar equation may be formed from the second two members of the above equation.

In determining the third and last element of exterior orientation, the survey azimuth $A_{z_{vo}}$ of the principal plane, it is convenient to orient the plate in the comparator so that the direction *vo* coincides with the positive *y* direction of the instrument and to set the initial readings on the coordinate scales

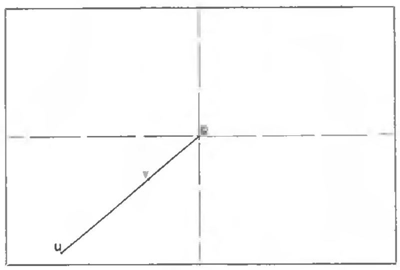

FIG. 85.

from measuring coordinates with respect to the origin *v*. This orientation, illustrated in Fig. 86, is accurately accomplished by using the point *u* marked on the line *ov* produced. With the plate in this position, coordinates are again read for the images *a*, *b*, and *c* of the control points *A*, *B*, and

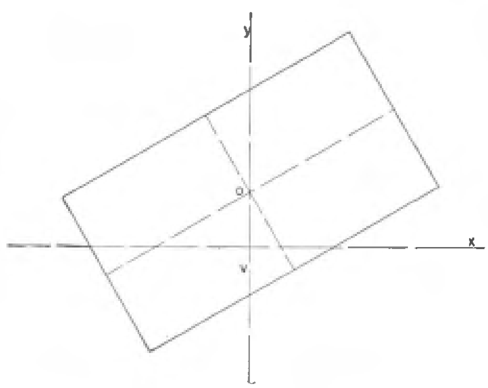

FIG. 86.

C. These coordinates will now be called $(x_a y_a)$, $(x_b y_b)$, and $(x_c y_c)$.

In Fig. 87, which shows the plate in its tilted position, the image *a* is shown for one of the three points. A line *aw* is drawn perpendicular to the principal line, *wk* is drawn perpendicular to *Lv*, and *k* is joined to *a*. Then *aw*, which is horizontal in space, is x_a and *vw* is y_a. Then

$$kw = y_a \cos t; \quad Lv = f \sec t; \quad kv = y_a \sin t$$

$$ka = \sqrt{x_a^2 + y_a^2 \cos^2 t}$$

and angle *kLa* is identical with angle m_A already computed. Then,

$$\left.\begin{aligned} \tan m_A &= \frac{\sqrt{x_a^2 + y_a^2 \cos^2 t}}{f \sec t - y_a \sin t} \\ \tan m_B &= \frac{\sqrt{x_b^2 + y_b^2 \cos^2 t}}{f \sec t - y_b \sin t} \\ \tan m_C &= \frac{\sqrt{x_c^2 + y_c^2 \cos^2 t}}{f \sec t - y_c \sin t} \end{aligned}\right\} \quad [16]$$

The three vertical angles computed by the equations above should agree exactly with those already

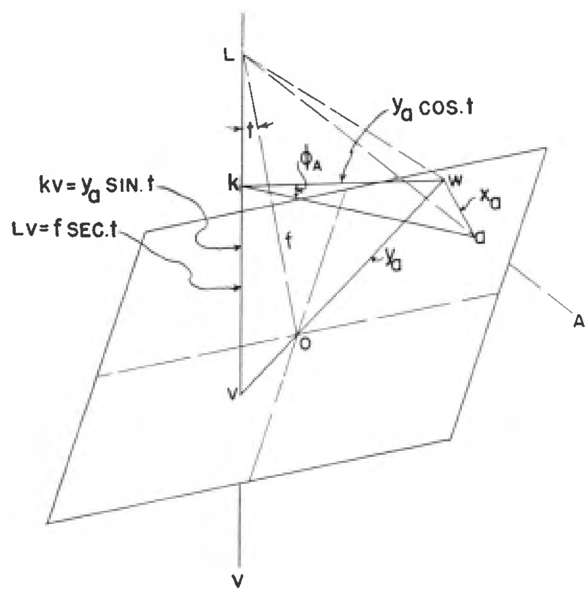

FIG. 87.

found by Eq. 10. This agreement furnishes a check on the determination of the nadir point and even on the determination of the exposure station.

The horizontal angle φ_A between the principal plane and the vertical plane through *LA* and *Lv*, as seen in the figure, can be found from

$$\left.\begin{aligned} \tan \varphi_A &= \frac{x_a}{y_a \cos t} \\ \tan \varphi_B &= \frac{x_b}{y_b \cos t} \\ \tan \varphi_C &= \frac{x_c}{y_c \cos t} \end{aligned}\right\} \quad [17]$$

The survey azimuth Az_{LA} of the vertical plane through LA, which is identical with the ground azimuth of VA, can be found exactly as in plane surveying. That is,

$$\left.\begin{aligned} \tan Az_{(LA)} &= \frac{x_L - x_A}{y_L - y_A} \\ \tan Az_{(LB)} &= \frac{x_L - x_B}{y_L - y_B} \\ \tan Az_{(LC)} &= \frac{x_L - x_C}{y_L - y_C} \end{aligned}\right| \quad [18]$$

From Fig. 87 the survey azimuth of the principal plane, $Az_{(vo)}$, can be obtained by subtracting φ_A from $Az_{(LA)}$. Therefore the third and last element of exterior orientation is found from

$$\left.\begin{aligned} Az_{(vo)} &= Az_{(LA)} - \varphi_A \\ Az_{(vo)} &= Az_{(LB)} - \varphi_B \\ Az_{(vo)} &= Az_{(LC)} - \varphi_C \end{aligned}\right] \quad [19]$$

The exact agreement between the three values furnishes another rigid check on the computation of the exterior orientation and on the computation of the exposure station.

The three elements of exterior orientation, together with the coordinates of the exposure station, completely fix the aerial photograph in space.

62. Determination of New Points from Comparator Measurements. With the photographs in a flight strip taken in the usual manner, with 60 per cent overlap in the direction of flight, every point on the ground will have its image appearing in at least two pictures. After the completion of the computation of the space coordinates of the exposure stations and the elements of exterior orientation for the two photographs in which images of a certain point appear, it is possible to compute from comparator measurements on these photographs the space coordinates of the new points referred to the ground or survey system of axes, that is, both its horizontal position and its elevation with respect to survey datum. This is the problem of *intersection in space.*

Place in the comparator (in turn) each plate upon which appears the image of an undetermined point p. It is most convenient in this part of the work again to orient the plate so that the positive motion of the comparator is in the direction of the principal line from v toward o, marked upon the plate, and to set the initial readings of the scales on the comparator so as to read coordinates based upon the nadir point v as the origin, as shown in Fig. 86 of the previous discussion. Designate by x, y the plate coordinates of the image p on the first plate, with respect to the origin v and the positive Y-axis vo. Use x', y' to indicate coordinates of the image p' on the second photograph, measured exactly the same.

In Fig. 88, which is like Fig. 87, except that the image a of a control point is replaced by the image

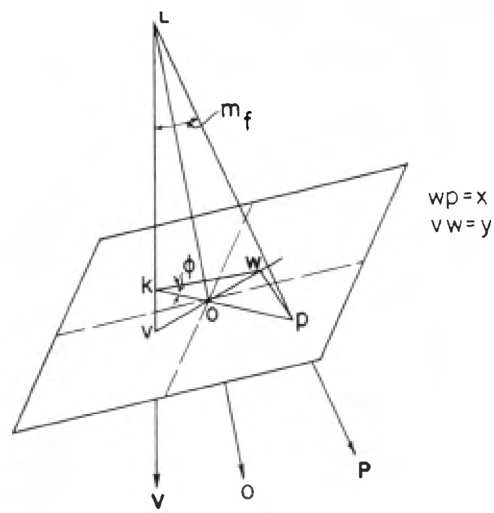

FIG. 88.

p, the undetermined point, wp, as before equals x and vw equals y. The horizontal angle wkp, designated by φ, being measured in a horizontal plane, is precisely the difference in azimuth between the principal plane and the vertical plane Lvp or LVP. Then,

$$\tan \varphi = \frac{x}{y \cos t} \text{ and } \tan \varphi' = \frac{x'}{y' \cos t'}$$

where t and t' are elements of the elements of the exterior orientations already determined.

Now if we add φ to $Az_{(vo)}$ the azimuth of the principal plane, another one of the elements of exterior orientation, we obtain the survey azimuth of the plane Lvp or LVP. This, in ordinary plane surveying, would be designated as the azimuth of LP or VP, which are identical since v is vertical beneath

L. Following the same system of notation therefore and calling this azimuth $Az_{(LP)}$, we have

$$Az_{(LP)} = Az_{(vo)} + \varphi$$

$$Az_{(L'P)} = Az'_{(v'o)} + \varphi'$$

The azimuths from the two known points L and L' to P now being known, we can calculate the horizontal position or the x and y survey coordinates of P by the ordinary intersection method of plane surveying.

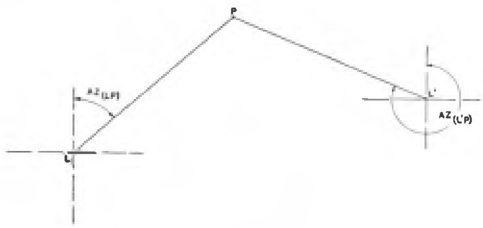

FIG. 89.

A very convenient method of finding x and y, the horizontal survey coordinates of P, is to follow the analytical geometry method of writing the equations of the horizontal projections of LP and $L'P$ and solving them simultaneously for the coordinates of the point of intersection. The slope s and s' of these two lines are numerically equal to cot $Az_{(LP)}$ and cot $Az_{(L'P)}$, respectively, and a positive sign is given the slope for a NE–SW line or a negative one for a NW–SE line. (See Fig. 89.)

The equations of the horizontal projections of LP and $L'P$, by the slope-point formula of analytics, are

$$y - y_L = s\,(x - x_L)$$

$$y - y_L' = s'(x - x_L')$$

The above linear equations can be solved simultaneously for x and y, giving directly the horizontal position of P. This is really easier than any of the usual methods of plane surveying.

It still remains to find z, the vertical coordinate of P, or its elevation above sea level or whatever xy-datum plane is used for the survey system of coordinates. This is done by the usual vertical angle method, as will be shown.

Call m the vertical angle vLp or VLP as measured by the first photograph (see Fig. 88) and m' the corresponding vertical angle as measured by the second photograph. These can be found from comparator coordinates, that is, from Fig. 88.

$$\tan m = \frac{kp}{Lk}$$

or

$$\tan m = \frac{\sqrt{x^2 + y^2 \cos^2 t}}{f \sec t - y \sin t}$$

and

$$\tan m' = \frac{\sqrt{x'^2 + y'^2 \cos^2 t'}}{f \sec t' - y' \sin t'}$$

Call L_H the projection of L upon a horizontal plane through the new point P as shown in Fig. 90.

$$L_HP = \sqrt{(x_L - x)^2 + (y_L - Y)^2}$$

then $\quad LL_H$ or $Z_L - Z = L_HP \cot m$

From which $\quad Z = Z_L - L_HP \cot m$

The value of Z can also be found from the value of m' by the similar equations.

The two values of Z thus found should check. The agreement in these values checks not only the calculations of the elevation but also the determination of the horizontal position.

The operations which we have just gone through are really directly analogous to the plane surveying methods for finding the horizontal position of a desired point by the intersection of measured horizontal directions from two known points, and then finding the elevation of a desired point by means of a vertical angle from either of the known points or from both for a check on the elevation and incidentally on the horizontal position.

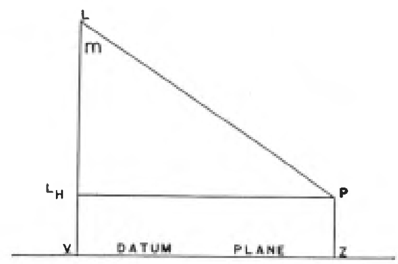

FIG. 90.

By the foregoing demonstration everything that can be accomplished in ordinary plane surveying can be accomplished by the use of photographs. Just as many points as are desired in the area of the survey can be determined horizontally and vertically. If a topographic map is desired, just as many planimetric details as are required can be accurately located, and just as many spot elevations as are desired can be determined, between which contours can be interpolated in the customary manner. Viewing the paper prints in a stereoscope will be found a valuable aid in choosing the critical points for which to determine spot elevations.

63. Curvature and Refraction in the Aerial Photograph. As far as the effect of refraction upon the problem of space intersection is concerned, the only error would be in the vertical angles, the angles designated throughout by m's, the horizontal angles called φ's being unaffected. The errors caused by refraction even in the vertical angles, however, within the vertical angle range of vertical photographs, are entirely negligible. Were highly precise work to be attempted by the use of oblique photographs it would be necessary to work out a practical method for taking the refraction into account. The effect of the curvature within the range of ordinary flight strips of vertical photographs, however, is far from negligible; in fact, it is not negligible within the area covered by a single photograph.

If the area being surveyed by means of the photographs does not exceed 40 or even 50 miles in linear dimensions, it will be found entirely satisfactory to use rectangular coordinates for all horizontal positions, that is for the x and y coordinates, exactly as described in the previous discussion. The effect of earth curvature will cause no trouble in the horizontal coordinates. It is in elevations that the curvature must be considered. In case a survey extends over 50 miles, it is beginning to assume such proportions that probably the control data will be stated in the form of geographic positions (latitudes and longitudes) instead of rectangular coordinates and probably the positions of points determined from the photographs will also be desired in this form.

It will now be shown that the effect of the earth's curvature can be determined as the curvature affects the determination of the elevation of a point. We will discuss it from the standpoint of a strip of pictures with three control points known in each picture.

In the methods discussed in the preceding pages the approximate space coordinates of an exposure station were designated by (x_L), (y_L), (z_L). These are determined at the outset by graphical method or otherwise. This is likewise the initial procedure for the case in hand.

While we are working with the first photograph we shall consider a plane of reference tangent to the surface of the earth at a point vertically underneath the true exposure station. This will be spoken of as the tangent plane for the first photograph.

If we consider the control points A, B, and C, as projected to this plane of reference, there will be

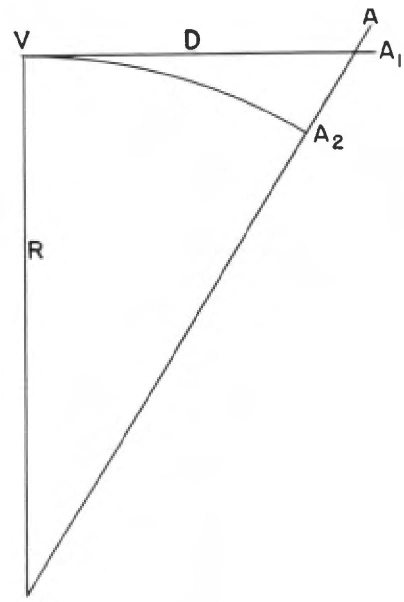

Fig. 91.

no appreciable change in their relative horizontal positions, and their x and d coordinates are used without alteration. But the elevations of A, B, and C above this datum plane will not be equal to their known elevations above sea level. In order to have these elevations referred to this datum plane, each one must be *decreased* by an amount equal to the curvature correction for the distance from the point in question to the point of tangency of the reference plane.

This curvature correction, shown in Fig. 91 as (A_1A_2), is found from the relation

$$(A_1A_2)\ (2R + A_1A_2) = D^2$$

to be $$A_1A_2 = \frac{D^2}{2R} \text{ approximately}$$

For practical purposes if D is in feet this correction in feet amounts to

$$0.0000000239D^2$$

The distance D for the point A, say, is found as accurately as it is needed for computing the curvature correction from

$$D = \sqrt{[(x_L) - x_A]^2 + [(y_L) - y_A]^2}$$

These corrections are computed and subtracted from the respective elevation z_A, z_B, and z_C. We will use H_A, H_B, and H_C to designate the elevations of these three control points referred to the tangent plane of the first photograph.

Using the coordinates (X_A, Y_A, H_A), (X_B, Y_B, H_B), (X_C, Y_C, H_C) for the three control points, we calculate the space coordinates of the exposure station L in precisely the same manner as described previously. It is to be noted that the elevation Z_L obtained for the exposure station, which refers to the tangent plane for the first photograph, likewise refers to sea level, for the reference plane is tangent at the point at sea level vertically beneath L.

The computation of the elements of exterior orientation is carried out in exactly the same manner as before.

This same method is followed in determining the space coordinates of the exposure station and the elements of exterior orientation in the second photograph, and in fact for each of the succeeding photographs. It is understood that a different tangent plane is used for each photograph, the angles between these planes being very small, amounting in fact to exactly the lengths of the terrestrial arcs between exposure stations.

It still remains for us to consider how to proceed with the calculation of new points by the method of intersection in space, that is, the modification of the previous computation to take into account the effect of the earth's curvature upon the elevation of new points.

In the problem of space intersection, the plate coordinates of the image of some new point P are measured on each of two photographs, in each case with respect to the nadir point v as the origin and with respect to a Y-axis coinciding with the line from v to the principal point o, as described before. The values of φ and φ' and of Az_{LP} and $Az_{L'P}$ are computed as before, and X and Y for the point P are found in the same way as before. The entire calculation of the horizontal position of P remains unchanged.

In fact the values of tan m and tan m' and of L_HP and $L_H'P$ are also found as before. But the formulas for computing the elevation of the new point are changed slightly, thus

$$z_L - H_P = L_HP \cot m \qquad H_P = z_L - (z_L - H_P)$$

and

$$z_L' - H_P' = L_H'P \cot m' \qquad H_P' = z_L' - (z_L' - H_P')$$

where H_P is the elevation of P above the tangent plane of one photograph and H_P' the elevation of P above the tangent plane of the other photograph.

To obtain z, the elevation of P above sea level, it is necessary to correct each of these values by *adding* the correction for curvature, Eq. 20.

$$C = 0.0000000239D^2 \qquad [20]$$

where D is the distance L_HP for correcting H_P and $L_H'P$ for correcting H_P'. The two values of z obtained by adding these corrections to Hp and Hp' should be equal.

For a more complete description of this method the reader is referred to a recent publication, "Analytical Computations in Aerial Photogrammetry" by Earl Church.