11. ELEMENTARY ALGEBRA. LOGARITHMS. PLANE GEOMETRY & TRIGONOMETRY.

11.1. Algebra.

This is similar to Arithmetic in so far as it is a Science dealing with numbers, but in Algebra letters are used to represent numbers.

Numbers having a fixed or known value are usually denoted by the letters a, b, c, etc.
Numbers whose values are unknown, or which have to be determined in any particular problem are usually denoted by x, y, or z.

The plus sign before a number means that number must be added, the minus sign before a number means that it must be subtracted from the number or numbers it is associated with; a+b means that b must be added to a. a-b means that b must be subtracted from a.

The sign X is used for multiplication. a X b is often written a.b or simply ab and means a must be multiplied by b.

The rules governing operations with signed numbers apply in algebra in exactly the same way as in arithmetic. Consequently one hears of "the algebraic sum" and the "algebraic difference"; meaning addition and subtraction in which the signs of the quantities are considered.

Conventions:-

a.a. is generally written a^2 and referred to as a squared.

a.a.a.------a where a appears n times is written a^n and referred to as a to the nth power.

A number which when squared is equal to a is called the square root of a and is written $\sqrt[2]{a}$ or simply $\sqrt{a}$.

A number which when cubed is equal to a is called the cube root of a and is written $\sqrt[3]{a}$

$\sqrt[n]{a}$ is the nth root of a.

Brackets.

Brackets are used to group numbers when several numbers have to be treated in the same way.

Thus (a + b)c means that a must be multiplied by c ,and b must be multiplied by c or a must be added to b and the result multiplied by c, and is equal to ac + bc.

Factors.

When some letter is common to all terms of an expression, each term and therefore the whole expression is divisible by that letter.

Thus:-

$$ac + bc = c(a + b)$$

$$by + \frac{b^2}{y} = b(y + \frac{b}{y})$$

$$3a^2x + 2aby = a(3ax + 2by).$$

EQUATIONS

In previous discussion of factors reference was made to the statement of equality:

$$a^2 - b^2 = (a - b)(a + b)$$

In this statement the two equal expressions $(a^2 - b^2)$ and $(a - b)(a + b)$ are called the sides of the equation. When, as in this case, the equality is true for every value of the letters involved, the equation is called an "identity". Sometimes an equality is only true for certain particular values of the letters involved and the name "equation" is generally reserved for such cases.

The letters a, b, c are used to denote known values and the letters x, y, z usually represent the unknown quantities which have to be found.

Solving an equation involves finding the values of the unknown quantities for which the equation is true. The values so found are said to satisfy the equation are are called the roots of the equation.

In the solution of equations, use is made of equivalent equations. Equations are equivalent when they have the same roots.

Equivalent equations may be formed by:

(1) Adding the same quantity to both sides for obviously

$x + c = y + c$ if and only if $x = y$.

(2) Transposing a term from one side of an equation to the other and changing its sign.

$x - y = o$ is equivalent to $x - y + y = y$ which is the same as $x = y$.

(3) Multiplying or dividing both sides by a quantity other than zero,

for obviously $mx = my$ is only true if $x = y$.

SIMPLE EQUATIONS

A simple equation involves only one unknown and the power to which the unknown is raised is one.

e.g. $$\frac{3X}{5} - 1 = \frac{X}{2} + 2.$$

This equation is solved by first ridding both sides of fractions by mutliplying by 10 giving $6x - 10 = 5x + 20$.

Then by transposing terms and changing signs we have

$$6x - 5x = 20 + 10$$

or $\underline{x = 30}$

Solve:

$$2b\ (x - a) + a(x + b) = a(a + b)$$

$$= 2bx - 2ab + ax + ab \quad = a^2 + ab.$$

Grouping terms containing x on the L.H.S. and transposing all others to the R.H.S.

$$2bx + ax = ab + a^2 + ab$$

$$x(a + 2b) = a(a + 2b)$$

$$\therefore \quad x = a.$$

All simple equations it is seen ultimately reduce to the form $ax + b = o$ from which $x = \frac{-b}{a}$.

If $b = o$ oviously $x = o$.

If b is not equal to zero but a = zero, then there are no finite values of x to satisfy the equation.

INDICES

$a.a$ is written a^2

$a.a.a$ is written a^3

$a.a.a$ ----- to n terms is written a^n

In a^2 2 is the index of a, in a^n n is the index of a.

$$a^n \times a^m = (a.a.a \text{ to n factors}).(a.a.a \text{ to m factors})$$

$$= (a.a.a.a \text{ to } (m + n) \text{ factors})$$

$$= a^{m + n}$$

Therefore $a^n.a^m = a^{m + n}$

Similarly $a^n.a^m.a^p.a^q = a^{m + n + p + q}$

Thus the index of the product of any number of powers of the same quantity is the sum of the indices of the factors.

Consider

$$(a^m)^n = a^m.\ a^m \text{ to n factors}$$

$$= a^{m + m + m \text{ to n terms}} = a^{mn}$$

$$\therefore\ (a^m)^n = a^{mn}$$

Value of $(a.b)^m$

$$(a.b)^m = ab \times ab \times ab \text{ to m factors}$$

$$= (a \times a \times a \text{ to m factors}).(b \times b \times b \text{ to m factors})$$

$$= a^m b^m$$

Similarly $(a.b.c.)^m = a^m b^m c^m$.

$$\therefore\ (a^x b^y c^z)^m = (a^x)^m\ (b^y)^m\ (c^z)^m$$

$$= a^{mx}\ b^{my}\ c^{mz}$$

So far it has been assumed that the indices used are positive whole numbers. Assuming that the same laws apply to all indices fractional and negative, meanings for such indices will now be determined.

Consider $a^{1/2}$

$$a^{1/2} \times a^{1/2} = a^{1/2 + 1/2} = a$$

i.e. $a^{1/2}$ multiplied by itself gives a, i.e. $a^{1/2} = \sqrt{a}$.

Similarly $a^{1/3} \times a^{1/3} \times a^{1/3} = a$

so $a^{1/3} = \sqrt[3]{a}$

and $a^{\frac{1}{n}} = \sqrt[n]{a}$

Consider $a^{\frac{2}{3}} = a^{1/3} \times a^{1/3}$

$= (a^2)^{1/3} = \sqrt[3]{a^2}$

$\therefore$ the denominator of the index indicates the root and the numerator of the index indicates the power

e.g. $(8)^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64}$

$= 4$

The meaning of a^o

$$a^m \times a^o = a^m$$

$$\therefore \quad a^o = \frac{a^m}{a^m}$$

$$= 1$$

e.g. $1^o = 1$, $10^o = 1$, $99^o = 1$ etc.

To find a meaning for a negative index

$a^m \times a^{-m} = a^{m-m} = a^o = 1$

$\therefore \quad a^m . a^{-m} = 1$

or $\quad a^{-m} = \frac{1}{a^m}$

$\therefore \quad a^{-m}$ is the reciprocal of a^m

$\therefore \quad a^2 . b^{-2} = \frac{a^2}{b^2}$

or $\quad \frac{a^2}{b^{-3}} = a^2 b^3$

11.2. Logarithms.

Definition

The logarithm of a number to a given base is the index of the power to which the base must be raised to equal the number.

If $a^x = N$, then x is called the logarithm of N to the base a, and the equation may be written

$$x = \log_a N,$$

$4^3 = 64 \quad \therefore \quad \log.\ 64 = 3.$ (to base 4)

$7^4 = 2401 \quad \therefore \quad \log.\ 2401 = 4.$ (to base 7)

Since $a^0 = 1 \quad \therefore \quad \log_a 1 = 0,$

$\therefore$ the logarithm of 1 to any base is 0.

For most practical purposes the base chosen is 10, and the logarithms are then called Common Logarithms; this system was introduced by Briggs in 1615. In writing down such logarithms the base is omitted, so that $\log_{10} 12$ is written log 12.

In many theoretical calculations the base used is the infinite series

$$1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots$$

which equals 2.7183 (approx.) and is denoted by e. Such logarithms are known as Napierian or Hyperbolic Logarithms.

We shall only deal with Common Logarithms.

RULE 1. The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

Let $\log m = x, \quad \therefore m = 10^x,$

$\log n = y, \quad \therefore n = 10^y,$

$\therefore mn = 10^x . 10^y = 10^{x+y},$

$\therefore \log (mn) = x + y$

$= \log m + \log n.$

This rule may be extended to the product of any number of factors, thus

$$\log (mnp) = \log m + \log n + \log p.$$

RULE 2. The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.

Let $\log m = x, \quad \therefore m = 10^x,$

$\log n = y, \quad \therefore n = 10^y,$

$\therefore \frac{m}{n} = \frac{10^x}{10^y} = 10^{x-y},$

$\therefore \log \left(\frac{m}{n}\right) = x - y$

$= \log m - \log n.$

RULE 3. The logarithm of the power of any number is equal to the logarithm of the number multiplied by the index of the power.

Let $\log m = x$

$\therefore\ m = 10^x,$

$m^n = 10^{nx},$

$\therefore\ \log (m^n) = nx$

$= n \log m.$

These rules have been proved for the base 10, but they would apply equally to any other base.

Given that log 7.211 = .8580 and log 8.878 = .9483,

Example 1: $\log (7.211 \times 8.878) = \log 7.211 + \log 8.878 = .8580 + .9483 = 1.8063.$

Example 2:
$$\log \frac{8.878}{7.211} = \log 8.878 - \log 7.211 = .9483 - .8580 = .0903.$$

Example 3:
$$\log (7.211)^5 = 5 \log 7.211 = 5 \times .8580 = 4.2900.$$

Example 4:
$$\log \sqrt[3]{7.211} = \tfrac{1}{3} \log 7.211 = \tfrac{1}{3} \times .8580 = .2860.$$

If a logarithm is partly integral and partly fractional, then the integral part is called the Characteristic and the fractional part the Mantissa.

It is always so arranged that the mantissa is positive.

Thus
$$\log \tfrac{1}{4} = \log 1 - \log 4$$
$$= 0 - .6021 \text{ (from Tables)}$$
$$= -1 + 1 - .6021$$
$$= -1 + .3979$$
$$= \bar{1}.3979.$$

If it is necessary to divide such a logarithm by a number, the negative characteristic is increased until it is a multiple of the divisor, compensation being made by adding the necessary positive integer.

Example. Given that $\log .03 = \bar{2}.4771$, find the value of $\log (.03)^{1/3}$.

$$\text{Log } (.03)^{1/3} = \tfrac{1}{3}\log .03 = \tfrac{1}{3} (\bar{2}.4771)$$
$$= \tfrac{1}{3} (\bar{3} + 1.4771)$$
$$= \bar{1}.4924.$$

$$10^3 = 1000 \quad \therefore \quad \log 1000 = 3,$$
$$10^2 = 100 \qquad \log 100 = 2,$$
$$10^1 = 10 \qquad \log 10 = 1,$$
$$10^0 = 1 \qquad \log 1 = 0,$$
$$10^{-1} = .1 \qquad \log .1 = -1,$$
$$10^{-2} = .01 \qquad \log .01 = -2,$$
$$10^{-3} = .001 \qquad \log .001 = -3.$$

It is therefore seen that the

Logarithm of a number between 100 and 1000, i.e. with 3 digits = 2 + fraction.

Logarithm of a number between 10 and 100, i.e. with 2 digits = 1 + fraction.

Logarithm of a number between 1 and 10, i.e. with 1 digit = 0 + fraction.

Logarithm of a number between .1 and 1 = $\bar{1}$ + fraction.

Logarithm of a number between .01 and .1 = $\bar{2}$ + fraction.

Logarithm of a number between .001 and .01 = $\bar{3}$ + fraction.

Thus the characteristic of the logarithm of a number can be written down by inspection.

RULE. The characteristic of the logarithm of a number greater than 1 is positive and is one less than the number of digits before the decimal point.

The characteristic of a number less than 1 is negative and is one more than the number of zeros immediately after the decimal point.

Consider the logarithm of the number 5.184. It must be between 0 and 1. From the tables log 5.184 = .7146.

Now consider

$$\log 51.84 = \log (10 \times 5.184) = \log 10 + \log 5.184 = 1.7146$$

$$\log 518.4 = \log (100 \times 5.184) = \log 100 + \log 5.184 = 2.7146$$

$$\log 5184 = \log (1000 \times 5.184) = \log 1000 + \log 5.184 = 3.7146$$

$$\log (.5184) = \log \frac{5.184}{10} = \log 5.184 - \log 10 = .7146 - 1 = \bar{1}.7146$$

$$\log (.05184) = \log \frac{5.184}{100} = \log 5.184 - \log 100 = .7146 - 2 = \bar{2}.7146$$

$$\log (.005184) = \log \frac{5.184}{1000} = \log 5.184 - \log 1000 = .7146 - 3 = \bar{3}.7146$$

It is seen that the logarithms of all numbers below 1 are negative, the log of (.5184) being .7146 - 1 = - .2854.

It is not, however, written in the form - .2854 but in the form $\bar{1}.7146$ where the mantissa or decimal part is always positive and the characteristic is written $\bar{1}$ (bar 1) to indicate that it is negative.

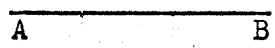
From the above it is seen that in the logarithm of a number the mantissa is not changed by changing the position of the decimal point. The characteristic of a number depends upon the position of the decimal point.

See Chambers Tables for a description of how to use the tables.

11.3. Plane Geometry and Trigonometry.

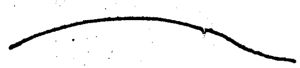
A line is the path of a point in motion; it has one dimension: length.

Thus, if a point is moved from the position A to the position B its path is the line AB



A straight line is a line that does not change its direction.

The distance between two points is the length of the straight line joining them.

A curved line  is a line that changes its direction at every point.

A broken line  is a line that changes its direction at certain points only. It is made up wholly of different straight lines.

The word line, when not qualified by any other word is understood to mean a straight line.

To produce a line is to prolong it or to increase its length.

A line can be produced to any extent in either direction.

To bisect any given magnitude is to divide it into two equal parts.

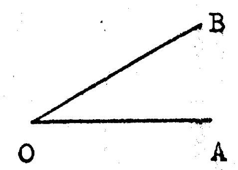
SIGNS AND ABBREVIATIONS

Sign	Meaning
$=$	is equal to or be equal to
$>$	greater than
$<$	less than
$\because$	because
$\therefore$	therefore
$\pm$	plus or minus
$\angle$	angle
$\triangle$	triangle

Sign	Meaning
$\perp$	perpendicular or perpendicular to
$\parallel$	parallel or parallel to
$\simeq$	approximately equal to

ANGLES AND PERPENDICULARS

An angle is the opening between two straight lines that meet in a point. The two straight lines are the sides, and the point where the lines meet is the vertex of the angle.

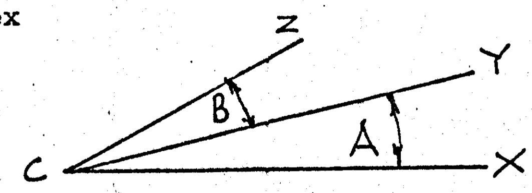


The straight lines OA and OB form an angle at the point O. The lines OA and OB are the sides of this angle, and the point O is its vertex.

An angle is referred to by naming a letter on each of its sides and a third letter at the vertex, the letter at the vertex being placed between the other two. Thus the angle in above figure is termed either

$\angle$ A O B or $\angle$ B O A

An angle may also be designated by a letter placed between its sides and near to the vertex



$\angle$ Z C Y or $\angle$ B $\qquad$ $\angle$ X C Y or $\angle$ A

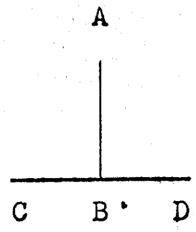
Two angles as shown above, having the same vertex and a common side are called adjacent angles.

Very often, letters from the Greek alphabet are used to designate angles

α	alpha	ϵ	epsilon	ι	iota	ν	nu	ρ	rho	ϕ	phi
β	beta	ζ	zeta	κ	kappa	ξ	xi	σ	sigma	χ	chi
γ	gamma	η	eta	λ	lambda	o	omicron	τ	tau	ψ	psi
δ	delta	θ	theta	μ	mu	π	pi	υ	upsilon	ω	omega

Two angles are equal when one can be placed on the other so that they will co-incide.

If a straight line meets another straight line so as to make with it two equal adjacent angles, each of these angles is a right angle (90°).

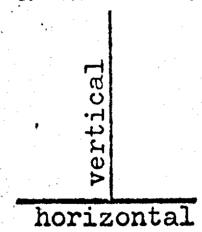


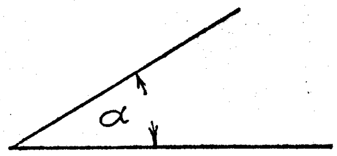
$\angle ABC = \angle ABD = 90^\circ$

The line A B is said to be perpendicular to the line C D. The point B (intersection of the two lines) is called the foot of the perpendicular.

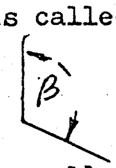
A horizontal line is a line parallel to the horizon, or to the surface of still water (liquid).

A vertical line is a perpendicular to a horizontal line and having, therefore, the direction of a plumb-line.

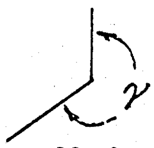


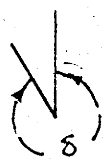


α is called an acute angle (less than a right angle).



β is called an obtuse angle (greater than one right angle but less than two).



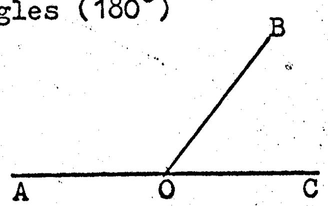


γ, δ are called reflex or re-entrant angles (greater than two right angles and less than four).

Any angle that is not a right angle is also spoken of as an oblique angle.

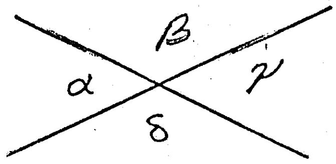
Two angles are said to be complementary when their sum is equal to one right angle.

Two angles are said to be supplementary when their sum is equal to two right angles (180°)



The sum of two adjacent angles whose non-common sides (AO and OC in above figure) are in the same straight line is equal to two right angles. $\angle$ A O B and $\angle$ B O C are supplementary.

Two intersecting straight lines determine four angles.



Any one of these angles and the angle on the opposite side of both lines, as the angles α and γ in above figure are called <u>vertical angles</u> with respect to each other, or <u>vertically opposite</u> angles. Vertical angles may also be defined as angles that have a common vertex and in which the sides of one are the prolongations of the sides of the other.

Since α and γ are each the supplement of β they are equal to each other.

Vertically opposite angles are equal.

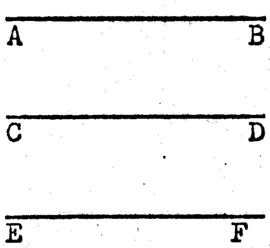
<u>In above figure</u>

$$\alpha + \beta = 180^\circ$$
$$\gamma + \beta = 180^\circ$$
$$\alpha = \gamma$$

<u>AXIOMS</u> (an <u>axiom</u> is a truth that is considered too obvious to require proof).

e.g. (i) The shortest distance between two points is along the straight line that joins them.

(ii) Straight lines that are parallel to the same straight line are parallel to one another.



$AB \parallel EF$

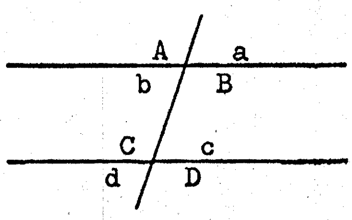
$CD \parallel EF$

$\therefore AB \parallel CD$

<u>PARALLELS</u>

Parallel lines are straight lines that lie in the same plane and never meet, however far they are produced.

Any two parallel lines have the same direction and are everywhere equally distant from each other. When two parallel lines are cut by a third line, the cutting line is called a secant line or transversal.



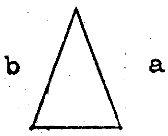
The eight angles thus formed are named as follows:

The angles a, A, d, D are exterior angles.
The angles b, B, c, C are interior angles
The pairs of angles a and d, and A and D are alternate-exterior angles.
The pairs of angles b and c, and B and C are alternate-interior angles.
The pairs of angles a and c, A and C, b and d, B and D are exterior-interior or corresponding angles.

Alternate-exterior angles are equal.
Alternate-interior angles are equal.
Exterior-interior (corresponding) angles are equal.

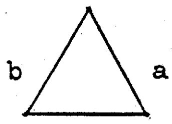
THE TRIANGLE

The triangles are named, according to their sides, as isosceles, equilateral and scalene triangles and according to their angles as right-angled and oblique-angled triangles.



a = b

isosceles



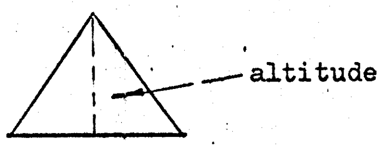
a = b = c

equilaterial triangle

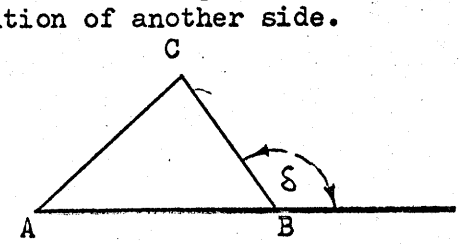
In a scalene triangle all three sides are of different length.

A right angled triangle is a triangle having one right angle. The side opposite the right angle is called hypotenuse.

The altitude of a triangle is the length of the line drawn perpendicular to the base (the side upon which the triangle is supposed to stand) from the vertex of the angle opposite to the base.

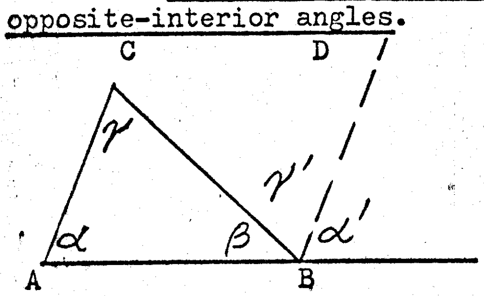


An exterior angle of a triangle is an angle formed by a side and the prolongation of another side.



δ is an exterior angle of ABC

In any triangle, an exterior angle is equal to the sum of the opposite-interior angles.



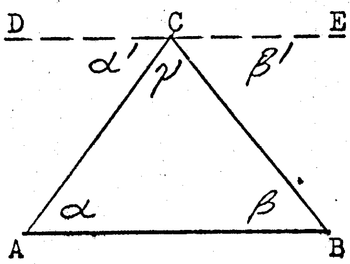
$AC \parallel BD$

$\gamma = \gamma'$ (alternate interior)

$\alpha = \alpha'$ (exterior-interior)

$\therefore \alpha + \gamma = \alpha' + \gamma'$

In any triangle the sum of the interior angles equals two right angles (180°).



$AB \parallel DE$

$\alpha = \alpha'$ (alternate-interior)

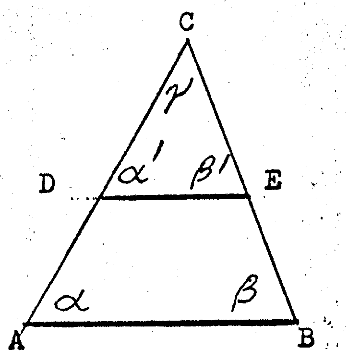
$\beta = \beta'$ (alternate-interior)

$\alpha' + \beta' + \gamma = 180^\circ$

$\therefore \alpha + \beta + \gamma = 180^\circ$

SIMILAR TRIANGLES

Similar triangles are those that have their corresponding angles equal and they have their sides about the equal angles, and consequently also those opposite to the equal angles, proportional.



$\triangle ABC$ similar $\triangle DEC$

γ (same angle for both)

$\alpha = \alpha'$) exterior-interior angles
$\beta = \beta'$)

(i) $AC : DC = AB : DE$

or $\frac{AC}{DC} = \frac{AB}{DE}$

(ii) $\frac{BC}{EC} = \frac{AB}{DE}$ etc.

A proportion is an equality of ratios or of fractions. Thus, the fractions $\frac{4}{5}$ and $\frac{8}{10}$ being equal, form a proportion.

In general, if $\frac{a}{b}$ is equal to $\frac{c}{d}$, these two ratios or fractions form a proportion, which may be written in any of the following forms:

$\frac{a}{b} = \frac{c}{d}$, $a : b = c : d$, $a : b :: c : d$

When written in either of the last two forms, the proportion is read "a is to b as c is to d".

The first and fourth terms of a proportion are called the extremes, the second and third, the means. Thus, in the proportion $a : b = c : d$, the extremes are a and d and the means are b and c.

The product of the extremes is equal to the product of the means.

$$a : b = c : d$$
$$a \times d = b \times c$$

The equation may be treated as any other algebraic equation. Both sides of the equation may be multiplied or divided by the same quantity, or the same quantity may be added to, or subtracted from both sides.

$$\frac{a}{b} = \frac{c}{d}$$

$$3.\frac{a}{b} = 3.\frac{c}{d}$$

$$2\frac{a}{b} = 2\frac{c}{d}$$

$$\frac{a}{b} + 3 = \frac{c}{d} + 3$$

$$\frac{a}{b} - 5 = \frac{c}{d} - 5$$

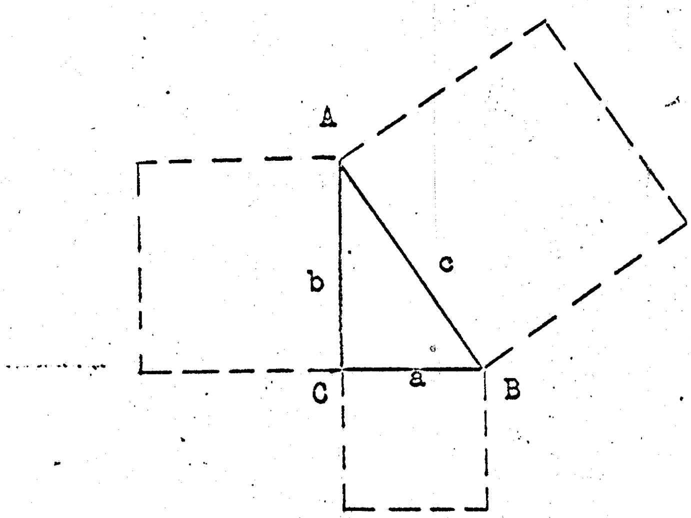
It is evident that if two fractions are equal, their reciprocals are also equal.

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{b}{a} = \frac{d}{c}$

If $\frac{8}{4} = \frac{16}{8}$ then $\frac{4}{8} = \frac{8}{16}$

Taking the reciprocal of a fraction is called <u>inverting</u> the fraction. The operation of inverting is called <u>inversion.</u>

<u>THEOREM OF PYTHAGORAS.</u>
In any right angled triangle a square described on the hypotenuse is equivalent to the sum of the squares described on the other two sides.

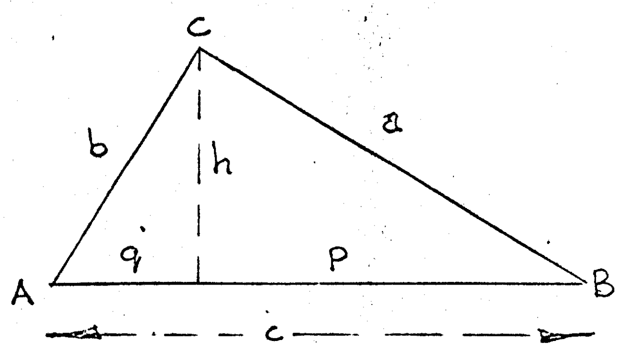


$$a^2 + b^2 = c^2$$

Theorem:

In a right angled triangle the square on the perpendicular to the Hypotenuse equals the rectangle contained by the two parts of the hypotenuse in which this is divided by the perpendicular.

$$h^2 = p\,q$$



Theorem:

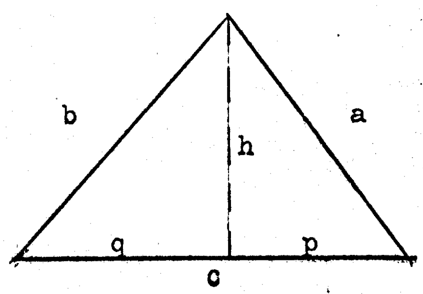
In a right angled triangle the square on one side equals the rectangle contained by the projection of that side on the hypotenuse and the hypotenuse.

$$a^2 = p\,c$$
$$b^2 = q\,c$$

Theorem:

In any triangle the projection of one side on the base equals the square of that side plus the square on the base less the square on the remaining side divided by twice the length of the base.

$$p = \frac{a^2 - b^2 + c^2}{2c} \qquad q = \frac{b^2 - a^2 + c^2}{2c}$$

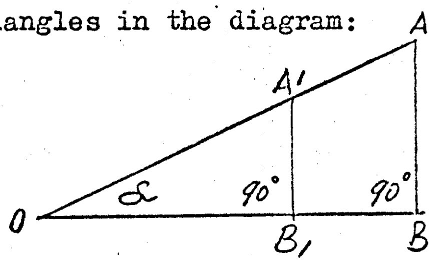


Trigonometrical Functions

It was explained previously that the corresponding sides of equiangular triangles are proportional.

Consider the equiangular right angled triangles in the diagram:

$$A_1B_1 : A_1O = AB : AO$$
$$A_1B_1 : OB_1 = AB : OB$$
$$OB_1 : OA_1 = OB : OA$$



These ratios are functions of angle α. No matter how far the foot of the perpendicular is from O the ratios remain constant. If α should change then the ratio will change. By tabulating these ratios for various angles we can determine the size of any acute angle provided the ratio is known, or determine the ratio if the angle is known. These ratios provide the means to computing unknown lengths or angles in triangles.

The names of these ratios are:

The ratio $\frac{\text{opposite side}}{\text{hypotenuse}}$ is called the "sine" of the angle considered

" " $\frac{\text{adjacent side}}{\text{hypotenuse}}$ " " " "cosine" " " " "

" " $\frac{\text{opposite}}{\text{adjacent side}}$ " " " "tangent" " " " "

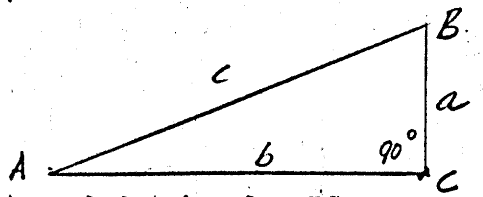
These three are the most important. If they are inverted, three other ratios are obtained:

the "cosecant" or $(\frac{1}{\text{sine}})$

the "secant" or $(\frac{1}{\text{cosine}})$

the "cotangent" or $(\frac{1}{\text{tangent}})$

As a general rule these ratios which, as defined, only apply to right angled triangles, are written in the abbreviated form: sin, cos, tan, cosec, sec and cot.



In the right angled triangle ABC:

$\sin A = \frac{\text{opposite to A}}{\text{hypotenuse}} = \frac{a}{c}$, $\sin B = \frac{b}{c}$

$\cos A = \frac{\text{adjacent to A}}{\text{hypotenuse}} = \frac{b}{c}$, $\cos B = \frac{a}{c}$

$\tan A = \frac{\text{opposite to A}}{\text{adjacent to A}} = \frac{a}{b}$, $\tan B = \frac{b}{a}$

$$\text{cosec } A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{a} \qquad \text{cosec } B = \frac{c}{b}$$

$$\text{sec } A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{b} \qquad \text{sec } B = \frac{c}{a}$$

$$\text{cot } A = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a} \qquad \text{cot } B = \frac{a}{b}$$

The angles A and B together add up to 90°; each being called the complement of the other; and it may be noticed that any ratio of one of the angles is equal to the co-ratio of its complement.
Hence the syllable "co" in cosine, cosec and cotan, is derived from the word "complement."

Thus sine A = co-sine of its complement B.
tan B = co-tan of its complement A.
sec A = co-sec of its complement B.

Descriptions of how to find the values of these functions for any angle are given in the table books but is essential to remember especially when interpolating that as the angle increases:

the sine increases
the cosine decreases

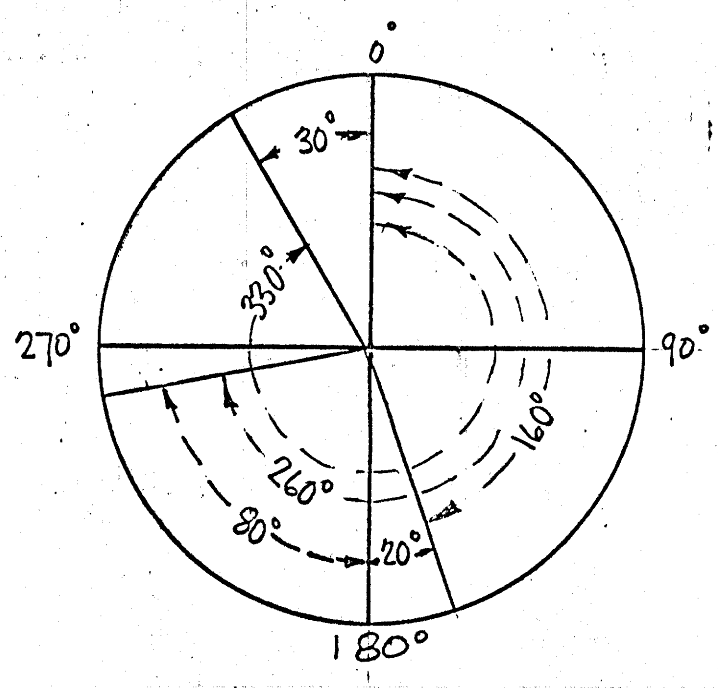
the tangent increases
the cotangent decreases

the secant increases
the cosecant decreases

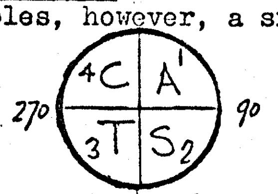
sin and cos are always less than 1
sec and cosec are always greater than 1
tan and cotan can be less than, equal to or greater than 1.

Angles greater than 90°

The trigonometrical functions listed in Chambers Tables are listed against angular values of from 0-90 degrees. However, the case where a function of an angle greater than 90° is required occurs as often as for angles less than 90°. Here is how the function is obtained:



Consider the circle divided into quadrants in the figure. If the function of an angle greater than 90° has to be found that angle has to be reduced to the 0°–180° line as a first step. Thus an angle of 160° reduced to give an acute angle with the 0°–180° line is worth 20°. The 260° angle reduces to 80°. The 330° angle reduces to 30°. The Rule: Subtract the angle from 180° or 360° or subtract 180° from the angle to obtain an ACUTE ANGLE. The result gives the angular value to be looked-up in the tables, however, a sign must be determined for the function.



The easiest way of remembering this is to visualize the above figure and recite the Sydney school boys mnemonic:

All	Stations	To	Central
All functions	Sines	Tangents	Cosines
1st Quadrant 0° – 90°	2nd Quadrant 90° – 180°	3rd Quadrant 180° – 270°	4th Quadrant 270° – 360°

are POSITIVE

and the same applies to the inverses – the cotangents, secants and cosecants.

Thus	sin	120°	is positive	+
	cos	120°	is negative	–
	sec	230°	is negative	–
	sec	340°	is positive	+
	cotan	245°	is positive	+
	cosec	320°	is negative	–
	cos	320°	is positive	+

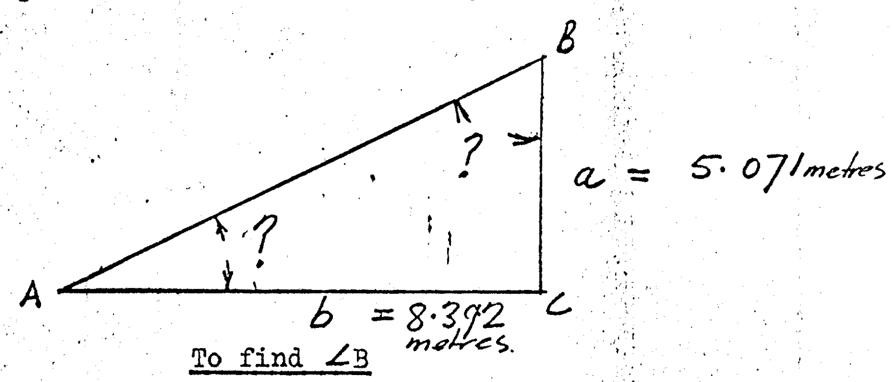
Solution of Plane Right Angle Triangles

A triangle has six components – 3 sides and 3 angles.

If three of these components are known the other 3 can be calculated.

Refer to the figure.

Example 1



To find ∠A

$\tan A = \frac{5.071}{8.392}$

$= 0.604266$

$A = 31° 08' 35''$

To find ∠B

$\cot B = \frac{5.071}{8.392}$

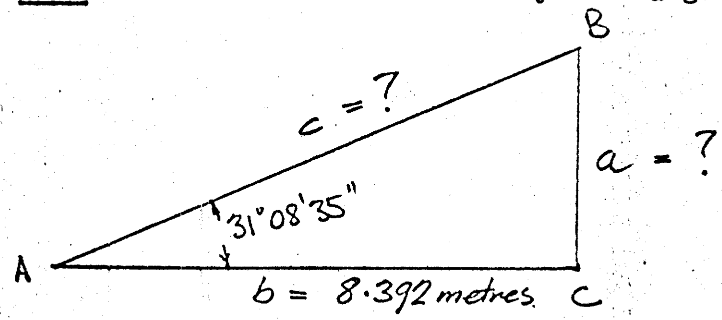
$= 0.604266$

$= 58° 51' 25''$

Check	A	$31^\circ\ 08'\ 35''$
	B	$58^\circ\ 51'\ 25''$
	c	$90^\circ\ 00'\ 00''$
Sum		$180^\circ\ 00'\ 00''$

Example 2

Note: Divide smaller distance by the larger.



To find 'a'

$\frac{a}{b} = \tan 31^\circ\ 08'\ 35''$

$a = b.\tan 31^\circ\ 08'\ 35''$

$= 8.392 \cdot 0.604266$

$= 5.071$ metres

To find 'c'

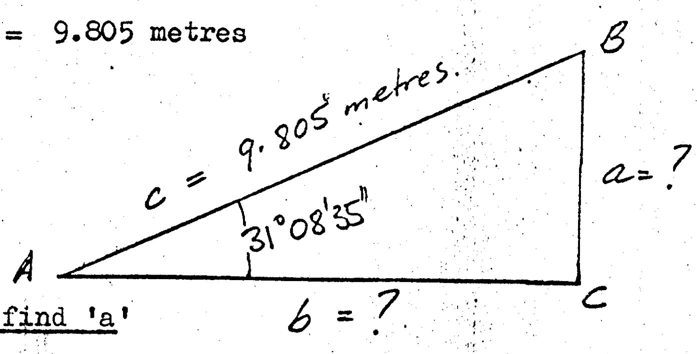
$\frac{c}{b} = \sec 31^\circ\ 08'\ 35''$

$c = b.\sec 31^\circ\ 08'\ 35''$

$= 8.392 \cdot 1.168390$

$= 9.805$ metres

Example 3



To find 'b'

$\frac{b}{c} = \cos 31^\circ\ 08'\ 35''$

$b = c \cdot \cos 31^\circ\ 08'\ 35''$

$= 9.805 \cdot 0.855879$

$= 8.392$ metres

To find 'a'

$\frac{a}{c} = \sin 31^\circ\ 08'\ 35''$

$a = c \cdot \sin 31^\circ\ 08'\ 35''$

$= 9.805 \cdot 0.517176$

$= 5.071$ metres

Solution of Oblique Angle Triangles

Case 1

Known:- Two sides and angle opposite one of these sides

Case 2

Known:- The angles and one side

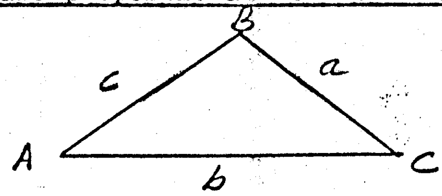
Case 3

Known:- Two sides and the angle included between them

Case 4

Known:- The three sides.

Cases 1 and 2 can be solved by the Sine Rule. This rule states: "The sides of any triangle are proportional to the sines of the angles opposite to them"



This rule can be expressed in mathematical notation as:

$$a : b : c = \sin A : \sin B : \sin C$$

or

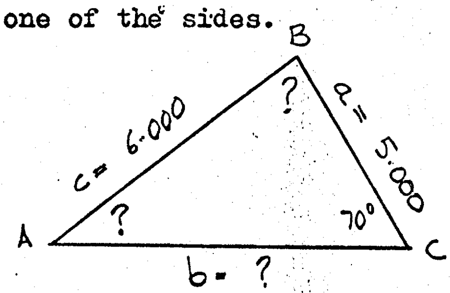
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{\sin B} = \frac{a}{b}, \quad \frac{\sin B}{\sin C} = \frac{b}{c}, \quad \sin \frac{C}{A} = \frac{c}{a}$$

Case 1:

Two known sides and the angle opposite one of the sides.



$$\frac{\sin A}{5.000} = \frac{\sin 70^\circ}{6.000}$$

$$\sin A = \frac{\sin 70^\circ \times 5.000}{6.000} = \frac{0.939693 \times 5.000}{6.000}$$

$$A = 51^\circ\ 32'\ 10'' = .78300$$

B may then be deduced by subtracting the sum of A & C from 180° thus:

$$\begin{aligned} \text{angle A} &= 51^\circ\ 32'\ 10'' \\ \text{angle C} &= 70^\circ\ 00'\ 00'' \\ \text{Sum} & \quad 121^\circ\ 32'\ 10'' \end{aligned}$$

$$\begin{aligned} & 180^\circ\ 00'\ 00'' \\ A + C \quad - \quad & 121^\circ\ 32'\ 10'' \\ \text{angle B} \quad & 58^\circ\ 27'\ 50'' \end{aligned}$$

Case 2:

Once B is known b can be computed.

$$\frac{b}{\sin 58^\circ\ 27'\ 50''} = \frac{6.000}{\sin 70^\circ\ 00'}$$

$$b = \frac{6.000 \sin 58^\circ\ 27'\ 50''}{\sin 70^\circ\ 00'} = 6.000 \sin 58^\circ\ 27'\ 50'' \operatorname{cosec} 70^\circ\ 00'$$

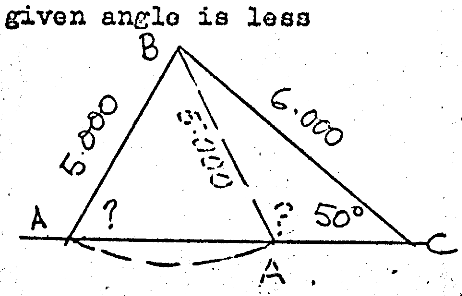
$$= 6.000 \times 0.852311 \times 1.064\ 178$$

$$= 5.442$$

The Ambiguous Case.

This occurs when the side opposite the given angle is less than the other given side.

Two solutions are possible.



Then A can be either acute or obtuse.

$$\frac{\sin A}{6.000} = \frac{\sin 50^\circ}{5.000}$$

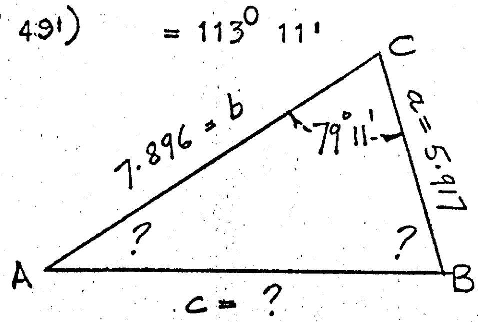
$$\sin A = \frac{\sin 50^\circ \cdot 6.000}{5.000}$$

$$= \frac{0.766044 \times 6.000}{5.000} = 0.919253$$

$$A = 66^\circ\ 49' \quad \text{or} \quad (180^\circ\ 00' - 66^\circ\ 49') = 113^\circ\ 11'$$

Case 3:

Two sides and included angle



$$\tan \frac{B - A}{2} = \frac{b - a}{b + a} \cdot \cot C/2$$

$$= \frac{7.896 - 5.917}{7.896 + 5.917} \cdot \cot C/2$$

$$= \frac{1.979}{13.813} \cdot \cot 39^\circ\ 35'\ 30'' \qquad = \frac{1.979}{13.813} \cdot 1.209\ 149$$

$$= 0.173\ 235 \qquad \frac{B - A}{2} = 9^\circ\ 49'\ 40''$$

$$B + A = 180^\circ - C = 100^\circ\ 49'$$

$$\frac{B + A}{2} = 50^\circ\ 24'\ 30''$$

$$\frac{B - A}{2} = 9^\circ\ 49'\ 40''$$

Sum = B = $60^\circ\ 14'\ 10''$

Difference A = $40^\circ\ 34'\ 50''$

C (given) = $79^\circ\ 11'\ 00''$

Sum (check) $180^\circ\ 00'\ 00''$

The summation only checks the final addition and subtraction. Nevertheless a handy check against blunders at that stage of the computation.

Once the angles are known the missing side can be computed by the Sine Rule. Note that there are two ways of doing this in the computation. By making this final calculation using both methods, the whole computation may be checked.

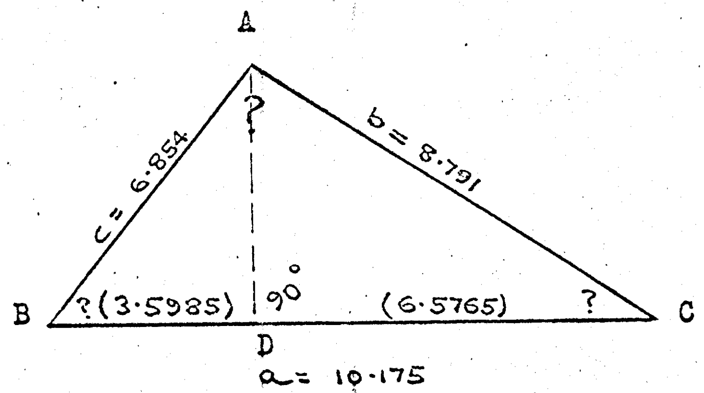
$$c = \frac{\sin 79^\circ\ 11' \ .\ 7.896}{\sin 60^\circ\ 40'\ 10''} = \frac{\sin 79^\circ\ 11' \ .\ 5.917}{\sin 40^\circ\ 34'\ 50''}$$

$$= \frac{0.982\ 233\ .\ 7.896}{0.868\ 079} = \frac{0.982\ 233\ .\ 5.917}{0.650\ 517}$$

$$= 8.934 = 8.934$$

Case 4.

The three sides known.



Let "a" be the greatest side and "b" greater than "c", then:-

$$CD - DB = \frac{(b-c)(b+c)}{a}$$

$$CD + DB = a$$

$$CD - DB = \frac{(8.791 - 6.854)\ (8.791 + 6.854)}{10.175} = \frac{1.937\ .\ 15.645}{10.175} = \frac{30.304}{10.175}$$

CD − DB = 2.978	CD − DB = 2.978	Check.
CD + DB = 10.175	CD + DB = 10.175	CD = 6.5765
2CD = sum = 13.153	2DB = diff = 7.197	DB = 3.5985
CD = ½sum = 6.5765	DB = ½diff = 3.5985	(sum) a = 10.1750

Angles B and C are now computed from the two right angled triangles ABD and ADC, as follows:-

$$\cos B = \frac{3.5985}{6.854} = 0.525\ 022 \qquad B = \underline{58^\circ\ 19'\ 50''}$$

$$\cos C = \frac{6.5765}{8.791} = 0.748\ 095 \qquad C = \underline{41^\circ\ 34'\ 30''}$$

Angle A is then found by deduction:-

B =	58° 19' 50"		180° 00' 00"
C =	41° 34' 30"	minus B+C	99° 54' 20"
B+C =	99° 54' 20"	A =	80° 05' 40"

<u>Check by Sine Rule:-</u>

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{10.175}{\sin 80^\circ\ 05'\ 40''} = \frac{8.791}{\sin 58^\circ\ 19'\ 50''} = \frac{6.854}{\sin 41^\circ\ 34'\ 30''}$$

$$\frac{10.175}{0.985\ 093} = \frac{8.791}{0.851\ 091} = \frac{6.854}{0.663\ 600}$$

$$10.329 = 10.329 = 10.329$$